NEW EXACT SOLUTIONS OF FRACTIONAL HIROTA-SATSUMA COUPLED KORTEWEG-DE VRIES EQUATIONS

by

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An improved extended tg-function method, which combines the fractional complex transform and the extended tanh-function method, is applied to find exact solutions of non-linear fractional partial differential equations. Generalized Hirota-Satsuma coupled Korteweg-de Vries equations are used as an example to elucidate the effectiveness and simplicity of the method.

Key words: *improved extended tg-function method, modified Riemann-Liouville derivative, fractional generalized Hirota-Satsuma coupled Korteweg-de Vries equation*

Introduction

For recent decades, there has been an increasing interest in the study of exact or approximate solutions of non-linear fractional differential equations that can be used to describe many phenomena in fluid mechanics, plasma physics, and heat conduction in porous media. Many effective methods have been proposed for fractional calculus, such as exp-function method [1], fractional sub-equation method [2], and so on.

However, to the best of our knowledge, it is still on a preliminary stage to search for exact analytical solutions of non-linear fractional differential equations. The motivation of this paper is to indicate how the exact solutions of non-linear fractional generalized Hirota-Satsuma coupled Korteweg-de Vries (KdV) equations can be constructed by the extended tg-function method. To achieve this aim, we recur to the well-known fractional complex transform method [3] and the extended tg-function method [4]. The fractional derivatives used in this present paper are in Jumarie's modified Riemann-Liouville sense.

Improved extended tg-function method

In this paper an auxiliary equation method based on the extended tg-function method [4] is improved, and is therefore called the improved extended tg-function method. The key steps of the improved extended tg-function method are provided.

Step 1. Consider the following non-linear space-time fractional differential equation:

$$P(u,\, D_t^{\alpha}u,\, D_x^{\beta}u,\, D_t^{2\alpha}u,\, D_x^{2\beta}u,\, \cdots) = 0, \qquad 0 < \alpha, \qquad \beta \le 1 \tag{1}$$

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where P is a polynomial in u and its various partial derivatives in which the highest order derivatives and non-linear terms are involved.

Step 2. By the fractional complex transform [3]:

$$\xi = \frac{x^\beta}{\Gamma(1+\beta)} - c\,\frac{t^\alpha}{\Gamma(1+\alpha)} \tag{2}$$

Equation (1) can be written into an ordinary differential equation (ODE):

$$P(u, -cu', u', c^2u'', u'', \cdots) = 0 \tag{3}$$

Step 3. Suppose that the solution of eq. (3) can be expressed as a polynomial in □:

$$u = \sum_{i=0}^{m} a_i \phi^i \tag{4}$$

where □ satisfies the Riccati equation in the form:

$$\phi'(\xi) = b + \phi^2(\xi) \tag{5}$$

Step 4. Substituting eq. (4) along with eq. (5) into eq. (3), get a set of algebraic equations about the parameters $a_0, \cdots, a_m$. Solving the obtained algebraic equations with the help of MATHEMATICA and taking advantage of the known solutions of eq. (5), we can obtain exact solutions of eq. (1) in concern.

For convenience, the solutions of eq. (5) are listed [4]:

$$\phi(\xi) = \begin{cases} -\sqrt{-b}\ \tanh(\sqrt{-b}\xi)\ \text{ or }\ -\sqrt{-b}\ \coth(\sqrt{-b}\xi), & b<0 \\ -\dfrac{1}{\xi}, & b=0 \\ \sqrt{b}\ \tan(\sqrt{b}\xi)\ \text{ or }\ \sqrt{b}\ \cot(\sqrt{b}\xi), & b>0 \end{cases} \tag{6}$$

The exact solutions of fractional generalized Hirota-Satsuma coupled KdV equations

Consider the following fractional generalized Hirota-Satsuma coupled KdV equations:

$$\begin{cases} D_t^\alpha u - \dfrac{1}{2} D_x^{3\alpha} u + 3uD_x^\alpha u - 3D_x^\alpha (vw) = 0 \\ D_t^\alpha v + D_x^{3\alpha} v - 3uD_x^\alpha v = 0 \\ D_t^\alpha w + D_x^{3\alpha} w - 3uD_x^\alpha w = 0, \qquad 0<\alpha\le 1 \end{cases} \tag{7}$$

By the fractional complex transform $\xi = (x^\alpha - ct^\alpha)/\Gamma(1+\alpha)$, eq. (7) can be converted into the ODE:

$$\begin{cases} -cu' - \dfrac{1}{2} u''' + 3uu' - 3(vw)' = 0 \\ -cv' + v''' - 3uv' = 0 \\ -cw' + w''' - 3uw' = 0 \end{cases} \tag{8}$$

By balancing the highest order derivative terms and non-linear terms in eq. (8), we suppose that:

$$u = a_0 + a_1\phi + a_2\phi^2, \qquad v = b_0 + b_1\phi + b_2\phi^2, \qquad w = c_0 + c_1\phi + c_2\phi^2 \tag{9}$$

Substituting eq. (9) along with eq. (5) into eq. (8) and taking advantage of Mathematica, yields the results:

$$a_0 = \frac{1}{3}(-c+8b), \quad a_1 = b_1 = c_1 = 0, \quad a_2 = 4, \quad b_0 = -\frac{4(3c_0 + 2cc_0 - 4c_2 b)}{3c_2^2}, \quad b_2 = \frac{4}{c_2} \tag{10}$$

Therefore, we get three types of quasi-traveling wave solutions of eq. (8):

$$\begin{aligned}
u_1 &= \frac{1}{3}(-c+8b) + 4\left[\sqrt{-b}\,\tanh\left(\sqrt{-b}\,\frac{x^\alpha - ct^\alpha}{\Gamma(1+\alpha)}\right)\right]^2 \\
v_1 &= -\frac{4(3c_0 + 2cc_0 - 4c_2 b)}{3c_2^2} + \frac{4}{c_2}\left[\sqrt{-b}\,\tanh\left(\sqrt{-b}\,\frac{x^\alpha - ct^\alpha}{\Gamma(1+\alpha)}\right)\right]^2 \\
w_1 &= c_0 + c_2\left[\sqrt{-b}\,\tanh\left(\sqrt{-b}\,\frac{x^\alpha - ct^\alpha}{\Gamma(1+\alpha)}\right)\right]^2 \\
u_2 &= \frac{1}{3}(-c+8b) + 4\left[\sqrt{-b}\,\coth\left(\sqrt{-b}\,\frac{x^\alpha - ct^\alpha}{\Gamma(1+\alpha)}\right)\right]^2 \\
v_2 &= -\frac{4(3c_0 + 2cc_0 - 4c_2 b)}{3c_2^2} + \frac{4}{c_2}\left[\sqrt{-b}\,\coth\left(\sqrt{-b}\,\frac{x^\alpha - ct^\alpha}{\Gamma(1+\alpha)}\right)\right]^2 \\
w_2 &= c_0 + c_2\left[\sqrt{-b}\,\coth\left(\sqrt{-b}\,\frac{x^\alpha - ct^\alpha}{\Gamma(1+\alpha)}\right)\right]^2
\end{aligned} \tag{11}$$

if $b < 0$ and c_0, c_2 are arbitrary constants that make the solutions eq. (11) meaningful.

$$\begin{aligned}
u_3 &= -\frac{c}{3} + 4\frac{\Gamma^2(1+\alpha)}{(x^\alpha - ct^\alpha)^2}, \qquad v_3 = -\frac{4(3c_0 + 2cc_0)}{3c_2^2} + \frac{4\Gamma^2(1+\alpha)}{c_2(x^\alpha - ct^\alpha)^2}, \\
w_3 &= c_0 + \frac{c_2\Gamma^2(1+\alpha)}{(x^\alpha - ct^\alpha)^2}
\end{aligned} \tag{12}$$

if $b = 0$ and c_0, c_2 are arbitrary constants that make the solutions eq. (12) meaningful.

$$u_4 = \frac{1}{3}(-c+8b) + 4\left[\sqrt{b}\,\tan\left(\sqrt{b}\,\frac{x^\alpha - ct^\alpha}{\Gamma(1+\alpha)}\right)\right]^2$$

$$v_4 = -\frac{4(3c_0 + 2cc_0 - 4c_2 b)}{3c_2^2} + \frac{4}{c_2}\left[\sqrt{b}\,\tan\left(\sqrt{b}\,\frac{x^\alpha - ct^\alpha}{\Gamma(1+\alpha)}\right)\right]^2$$

$$w_4 = c_0 + c_2\left[\sqrt{b}\,\tan\left(\sqrt{b}\,\frac{x^\alpha - ct^\alpha}{\Gamma(1+\alpha)}\right)\right]^2$$

(13)

$$u_5 = \frac{1}{3}(-c+8b) + 4\left[\sqrt{b}\,\cot\left(\sqrt{b}\,\frac{x^\alpha - ct^\alpha}{\Gamma(1+\alpha)}\right)\right]^2$$

$$v_5 = -\frac{4(3c_0 + 2cc_0 - 4c_2 b)}{3c_2^2} + \frac{4}{c_2}\left[\sqrt{b}\,\cot\left(\sqrt{b}\,\frac{x^\alpha - ct^\alpha}{\Gamma(1+\alpha)}\right)\right]^2$$

$$w_5 = c_0 + c_2\left[\sqrt{b}\,\cot\left(\sqrt{b}\,\frac{x^\alpha - ct^\alpha}{\Gamma(1+\alpha)}\right)\right]^2$$

if $b > 0$ and c_0, c_2 are arbitrary constants that make the solutions eq. (13) meaningful.

As α approaches to 1, eqs. (11)-(13) become exact traveling wave solutions of standard form of Hirota-Satsuma coupled KdV equations. These fractional solutions cannot be obtained by other approximate methods.

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