EFFICIENT HOMOTOPY PERTURBATION METHOD FOR FRACTIONAL NON-LINEAR EQUATIONS USING SUMUDU TRANSFORM

by

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In this paper, we propose an efficient modification of the homotopy perturbation method for solving fractional non-linear equations with fractional initial conditions. Sumudu transform is adopted to simplify the solution process. An example is given to illustrate the solution process and effectiveness of the method.

Key words: *approximate solution, homotopy perturbation method, Sumudu transform, Jumarie's modified R-L fractional derivative*

Introduction

In the past few decades, fractional calculus [1] has been widely used in anomalous diffusion, viscoelastic fluid, thermal conduction, control, turbulence, etc, and the corresponding fractional differential equations have been solved by a wide class of methods, for examples, analytical methods [2], numerical methods, and some semi-analytical techniques, such as variational iteration method [3], and homotopy perturbation method [4, 5]. Khan *et al.* [6] proposed a new technique for solving fractional equations with fractional initial conditions. Recently, Sumudu transform has been used as a valuable tool to solve fractional systems [7]. Motivated and inspired by thinking of Khan *et al.* [6], we give a new modification of homotopy perturbation method which is based on Sumudu transform. In this work, we will use this method to solve fractional non-linear heat-like equations with fractional initial conditions.

Fundamental properties of fractional calculus and Sumudu transform

Jumarie's modified R-L calculus theory

Definition 1. Let $f: R \to R,\ t \to f(t)$ denote a continuous but not necessarily differentiable function. Then its fractional derivative of order α, $0 < \alpha < 1$ is defined by:

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$$f^{(\alpha)}(t)=[f^{(\alpha-1)}(t)]'=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-\xi)^{-\alpha}[f(\xi)-f(0)]\,\mathrm{d}\xi \tag{1}$$

where $n\le\alpha\le n+1,\quad n\ge1,\quad f^{(\alpha)}(t)=[f^{(\alpha\text{-n})}(t)]^n$. The Laplace transform $L\{\cdot\}$ of the fractional derivative is $L[f^{(\alpha)}(t)]=s^{\alpha}L[f(t)]-s^{\alpha-1}f(0),\ 0<\alpha<1$.

Proposition 1. Let α be such that $0 < \alpha < 1$, there are two different ways [8] to obtain $D^{3\alpha}f(t)$, one can calculate $D^{\alpha}D^{\alpha}D^{\alpha}f(t)$ to obtain the following Laplace transform:

$$L[D^{\alpha}D^{\alpha}D^{\alpha}f(t)]=s^{3\alpha}f(s)-s^{3\alpha-1}f(0)-s^{2\alpha-1}f^{(\alpha)}(0)-s^{\alpha-1}f^{(2\alpha)}(0) \tag{2}$$

Sumudu transform

Definition 2. The Sumudu transform of a function $f(t)$, defined for all real numbers $t\ge0$, is the function $F(u)$, defined by $F(u)=S[f(t)]=\int_0^{\infty}f(ut)\mathrm{e}^{-t}\mathrm{d}t$.

Proposition 2. The Sumudu transform [8] of the derivatives with integer order is $S[\mathrm{d}f(t)/\mathrm{d}t] = 1/u\ [F(u) - F(0)]$. We can also get the Sumudu transform of the n-order derivative as:

$$S\left[\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right]=\frac{1}{u^n}\left[F(u)-\sum_{k=0}^{n-1}u^k\,\frac{\mathrm{d}^k f(t)}{\mathrm{d}t^k}\Big|_{t=0}\right]$$

Sumudu transform of modified R-L fractional calculus

Proposition 3. For a positive integer n and $0 < \alpha \le 1$, there are different ways to obtain $D^{n\alpha}f(t)$, one can calculate $\underbrace{D^{\alpha}\cdots D^{\alpha}}_{n}f(t)$ to obtain the Sumudu's transform modified R-L fractional calculus

$$S[\underbrace{D^{\alpha}\cdots D^{\alpha}}\,f(t)]=\frac{1}{u^{n\alpha}}F(\mathrm{u})-\sum_{k=0}^{n-1}\frac{1}{u^{(\mathrm{n-k})\alpha}}f^{(\mathrm{k}\,\alpha)}(0)$$

where $D^{n\alpha}=\partial^{n\alpha}/\partial t^{n\alpha}$ is the $n\alpha$ order modified R-L fractional derivative.

Description of the method

In order to elucidate the solution procedure of the modified homotopy perturbation method, we consider the following general non-linear system:

$$D^{n\alpha}U(x,t)=R[U(x,t)]+N[U(x,t)]+g(x,t) \tag{3}$$

where $k = 0, \dots n - 1$, $U^{(k\alpha)}(x, 0^+) = a_k$, $g(x, t)$ is the source term, N – the general non-linear differential operator, and R – the linear differential operator. In view of homotopy perturbation method [4], we can construct a homotopy for eq. (3):

$$D^{n\alpha}U(x,t)=p\{R[U(x,t)]+N[U(x,t)]+g(x,t)\}$$

where $p\in[0, 1]$ is an embedding parameter. If $p = 0$, the equation becomes $D^{n\alpha}U(x, t) = 0$, and when $p = 1$, it turns out to be the original fractional differential equation. In the homotopy perturbation method, the basic assumption is that the solutions can be written as a power series in p:

$$U(x, t) = \sum_{n=0}^{\infty} p^n U_n(x, t)$$

and the non-linear term can be decomposed as:

$$NU(x, t) = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(U)$$

where $\mathcal{H}_n(U)$ is He's polynomials [5], which can be generated by:

$$\mathcal{H}_n(U_0, \cdots, U_n) = \frac{1}{n\,!}\frac{\partial^n}{\partial p^n}\left(\mathrm{N}\sum_{i=0}^{n} p^i U_i\right)_{p=0}, \qquad n = 0, 1, 2, ...$$

According to the homotopy perturbation method, and collecting the terms with the same powers of p, we can obtain a series of equations of the form:

$$\begin{cases} p^0: & D^{n\alpha}U_0(x, t) = 0 \\ p^1: & D^{n\alpha}U_1(x, t) = RU_0(x, t) + \mathcal{H}_0(U) + g(x, t) \\ p^2: & D^{n\alpha}U_2(x, t) = RU_1(x, t) + \mathcal{H}_1(U) \\ & \vdots \end{cases} \tag{4}$$

by means of **Proposition 3** given in eq. (3), we get:

$$\begin{cases} p^0: & \dfrac{1}{u^{n\alpha}}U_0(x, u) - \displaystyle\sum_{k=0}^{n-1}\frac{1}{u^{(\mathrm{n-k})\alpha}}\,U^{n\alpha}U_0(x, 0) = 0 \\ p^1: & \dfrac{1}{u^{n\alpha}}U_1(x, u) = RU_0(x, u) + S[\mathcal{H}_0(U)] + g(x, u) \\ p^2: & \dfrac{1}{u^{n\alpha}}U_2(x, u) = RU_1(x, u) + S[\mathcal{H}_1(U)] \\ & \vdots \end{cases} \tag{5}$$

Solving for U_0, U_1, U_2, …, respectively, and using the fractional initial value conditions, we obtain:

$$\begin{cases} p^0: & U_0(x, t) = S^{-1}\left[\displaystyle\sum_{k=0}^{n\text{-}1} u^{ka}U^{ka}U_0(x, 0)\right] \\ p^1: & U_1(x, t) = S^{-1}\left\{u^{na}\{RU_0(x, u) + S[H_0(U)] + g(x, u)\}\right\} \\ p^2: & U_2(x, t) = S^{-1}\left\{u^{na}\{RU_1(x, u) + S[H_1(U)]\}\right\} \\ & \vdots \end{cases} \tag{6}$$

Substituting successive iterations in $U(x, t) = \sum_{n=0}^{\infty} U_n(x, t)$, will give the required result.

Modified R-L fractional equations

Example 1. Consider the following one-dimensional fractional non-linear heat-like equation:

$$D^{2\alpha}U(x,t)=x^2(U_xU_{xx})_x-x^2(U_{xx})^2-U(x,t) \tag{7}$$

where $U(x,0)=0,\ U^{(\alpha)}(x,0)=x^2,$ and $t\geq 0,\ 0\leq x\leq 1,\ 0<\alpha\leq 1.$ Using the procedure in the section *Description of the method*, we can get the form:

$$\begin{cases} U_0(x,t)=S^{-1}\{u^\alpha x^2\} \\ U_1(x,t)=S^{-1}\left\{u^{2\alpha}\{x^2S[\mathcal{H}_{10}(U)-\mathcal{H}_{20}(U)]-U_0(x,u)\}\right\} \\ U_2(x,t)=S^{-1}\left\{u^{2\alpha}\{x^2S[\mathcal{H}_{11}(U)-\mathcal{H}_{21}(U)]-U_1(x,u)\}\right\} \\ \quad\vdots \end{cases} \tag{8}$$

where $\mathcal{H}_{1n}$ and $\mathcal{H}_{2n}$ are He's polynomials that represent the non-linear term $(U_xU_{xx})_x$ and $(U_{xx})^2$, respectively, we have a few terms of the He's polynomials for $(U_xU_{xx})_x$ and $(U_{xx})^2$ which are given by

$$\mathcal{H}_{10}=U_{0xx}U_{0xx}+U_{0x}U_{0xxx},\ \cdots, \text{ and}$$

$$\mathcal{H}_{20}=U_{0xx}U_{0xx},\ \mathcal{H}_{21}=U_{0xx}U_{1xx}+U_{1xx}U_{0xx},\ \cdots$$

taking the previous method, eq. (8) in the most refined form can be written as:

$$U_1(x,t)=-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}x^2,\qquad U_0(x,t)=\frac{t^\alpha}{\Gamma(\alpha+1)}x^2,\qquad U_2(x,t)=\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}x^2,\cdots$$

the series solution is:

$$U(x,t)=x^2\left(\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}-\frac{t^{7\alpha}}{\Gamma(7\alpha+1)}+\cdots\right)$$

If $\alpha = 1$, we obtain:

$$U(x,t)=x^2\left(t-\frac{t^3}{3!}-\frac{t^5}{5!}+\frac{t^7}{7!}+\cdots\right)=x^2\sin(t)$$

which is the exact solution of this case.

Conclusion

In this paper, fractional non-linear equations with fractional initial conditions are solved by using Sumudu transform. The technique provides a certain value for other fractional initial condition problems. The results of introducing fractional order initial conditions and the Sumudu transform for the studied cases show the high accuracy, simplicity, and efficiency of the technique.

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