HE'S FRACTIONAL DERIVATIVE FOR HEAT CONDUCTION IN A FRACTAL MEDIUM ARISING IN SILKWORM COCOON HIERARCHY

by

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He's fractional derivative is adopted in this paper to study the heat conduction in fractal medium. The fractional complex transformation is applied to convert the fractional differential equation to ordinary different equation. Boltzmann transform and wave transform are used to further simplify the governing equation for some special cases. Silkworm cocoon are used as an example to elucidate its natural phenomenon.

Key words: *fractional differential equation, fractional complex transform*

Introduction

There are many definitions on fractional derivatives. Here are some definitions. Caputo fractional derivative is defined as [1]:

$$D_x^\alpha[f(x)] = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n} \mathrm{d}t \tag{1}$$

Riemann-Liouville fractional derivative reads [1]:

$$D_x^\alpha[f(x)] = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\mathrm{d}t \tag{2}$$

Jumarie's modification of Riemann-Liouville fractional derivative is [2]:

$$D_x^\alpha[f(x)] = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x (x-t)^{n-\alpha-1}[f(t)-f(0)]\mathrm{d}t \tag{3}$$

Xiao-Jun Yang's fractional derivative gives [1]:

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$$D_x^{(\alpha)} f(x_0) = f^{(\alpha)}(x_0) = \frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}\Big|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha [f(x) - f(x_0)]}{(x-x_0)^\alpha} \tag{4}$$

where $\Delta^\alpha[f(x) - f(x_0)] \cong \Gamma(1+\alpha)\Delta[f(x) - f(x_0)]$.

Ji-Huan He's fractional derivative is defined as [3]:

$$D_t^\alpha f = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1} [f_0(s) - f(s)]\mathrm{d}s \tag{5}$$

where $f_0(x)$ is a known function.

In this paper He's definition [3] is adopted to study the heat conduction in a fractal porosity.

Heat transfer in fractal medium

The Fourier's Law of thermal conduction in a fractal medium can be expressed as [1]:

$$q = -D \frac{\partial^\alpha T}{\partial x^\alpha} \tag{6}$$

and

$$\frac{\partial T}{\partial t} = -\frac{\partial^\alpha q}{\partial x^\alpha} \tag{7}$$

where q is the heat flux, T – the temperature, D – the thermal conductivity of heat flux in the fractal medium, and $\partial^\alpha/\partial x^\alpha$ – the He's fractional derivative defined as [3]:

$$\frac{\partial T^\alpha}{\partial x^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_{t_0}^{t} (s-x)^{n-\alpha-1} [T_0(s) - T(s)]\mathrm{d}s \tag{8}$$

where $T_0(x)$ can be the solution of its continuous partner of the problem with the same boundary/initial conditions of the fractal partner.

From both eqs. (6) and (7), we can derive the fractional heat conduction equation, which shows:

$$\frac{\partial T}{\partial t} = \frac{\partial^\alpha}{\partial x^\alpha}\left(D \frac{\partial^\alpha T}{\partial x^\alpha}\right) \tag{9}$$

where α is the fractional dimension of the fractal medium.

The fractional complex transformation [4-6] can convert a fractional differential equation to a partial differential equation. By the fractional complex transformation [4-6]:

$$s = \frac{x^\alpha}{\Gamma(1+\alpha)} \tag{10}$$

Equation (9) is converted to a partial differential equation, which reads:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial s}\left(D \frac{\partial T}{\partial s} \right) \tag{11}$$

Introducing the Boltzmann variable λ defined as:

$$\lambda = \frac{s}{\sqrt{t}} = \frac{x^{\alpha}}{\sqrt{t}\ \Gamma(1+\alpha)} \tag{12}$$

We can convert eq. (11) into the following ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(D \frac{\mathrm{d}T}{\mathrm{d}\lambda} \right) + \frac{\lambda}{2}\frac{\mathrm{d}T}{\mathrm{d}\lambda} = 0 \tag{13}$$

Equation (13) can be solved by the variational iteration method, the homotopy perturbation methods and others. A complete review on various analytical methods is available in [7].

If a wave solution of eq. (11) is searched for, we can introduce the transform:

$$T(s,t) = T(\xi), \qquad \xi = s - kt = \frac{x^{\alpha}}{\Gamma(1+\alpha)} - kt \tag{14}$$

Equation (11) becomes:

$$k\frac{\mathrm{d}T}{\mathrm{d}\xi} + \frac{\partial}{\partial \xi}\left(D \frac{\partial T}{\partial \xi} \right) = 0 \tag{15}$$

Solving eq. (15) by integration, we obtain:

$$T(\xi) = c_1 + c_2 \exp\left(-\frac{k\xi}{D} \right) \tag{16}$$

where c_1 and c_2 are integral constants.

Substituting eq. (14) into eq. (16) results in a general exact solution:

$$T(x,t) = c_1 + c_2 \exp\left(\frac{k^2 t}{D} - \frac{kx^{\alpha}}{D\Gamma(1+\alpha)} \right) \tag{17}$$

As an example, we consider heat conduction in cocoon, which is considered as a fractal medium. As you know, in continuous media, heat transfer of an isotropic medium without inner heat source can be regarded as one-dimensional steady case, that is:

$$D\frac{\mathrm{d}T}{\mathrm{d}x} = Q \tag{18}$$

The solution of eq. (18) is:

$$T = T_0 + \frac{Q}{D}x \tag{19}$$

where T_0 is the initial temperature at $x = 0$ and Q – the heat flux.

However, heat transfer in the discontinuous hierarchic silkworm cocoon can be expressed as:

$$D\frac{\partial\overline{T}}{\partial s}=\overline{Q} \tag{20}$$

Integrating eq. (20) results in:

$$\overline{T}=\overline{T}_0+\frac{\overline{Q}}{D}\frac{x^{\alpha}}{\Gamma(1+\alpha)} \tag{21}$$

where $\overline{T}_0$ is the initial heat flux concentration, $\overline{T}$ – the heat flux concentration, and $\overline{Q}$ – the heat flux in the hierarchic silkworm cocoon.

Suppose the heat flux of hierarchic silkworm cocoon is equal to that of continuous media, that is, $\overline{Q}=Q$. It is obvious that:

$$\frac{\overline{T}-\overline{T}_0}{T-T_0}=\frac{x^{\alpha-1}}{\Gamma(1+\alpha)} \tag{22}$$

Discussions

Due to continuous media, $\overline{T}-\overline{T}_0 = = T-T_0$ in the eq. (22). Suppose the fractional dimension α = 0.618, then x = 1.33. x represents the smallest mearsuring size between layer and layer, shown in fig. 1.

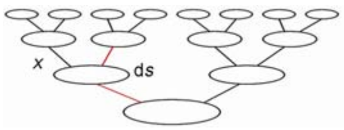

Figure 1. Schematic diagram of hierarchic heat transfer path in cocoon

Table 1. The relation between x and heat flux in hierarchic silkworm cocoon

x	100	1	1.33	10^{-2}	10^{-4}	10^{-6}	10^{-8}
$\frac{\overline{T}-\overline{T}_0}{T-T_0}$	0.19	1.11	1	6.48	37.6	218.6	1269.7

For the hierarchic silkworm cocoon, also suppose the fractional dimensions α = 0.618. The various values of $(\overline{T}-\overline{T}_0)/$ $/(T-T_0)$ are given in tab. 1.

As can be seen, when $x < 10^{-4}$, the nanoeffect is very significant. Especially, when x reaches 10 nm, heat transfer property of fractal media is so much unusual compared with that of continuous media. So the structure feature of cocoon provides us with an optimal template, which could be duplicated in biomimic fabric design to improve the heat-moisture comfort ability of an apparel.

Conclusions

He's fractional derivative follows chain rule [3], and the fractional complex transformation can convert the fractional differential equation into partial differential equation, which can be further simplified into ordinary differential equation by either Boltzmann transform or wave transform, so that we can adopt all analytical methods or numerical methods developed for ordinary differential equations can be applied for fractional calculus.

The fractal model for heat transfer in hierarchic porous cocoons has been proposed for the first time based on the local fractional calculus theory. Moreover, the nanoeffect of this model for heat transfer has been proved to be very significant at nanoscales. Therefore, it would be helpful for biomimic design in many fields, such as biomaterials, functional textiles and aviation industry.

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