UNSTEADY FREE CONVECTION FLOW OF A MICROPOLAR FLUID WITH NEWTONIAN HEATING: CLOSED FORM SOLUTION

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This article investigates the unsteady free convection flow of a micropolar fluid over a vertical plate oscillating in its own plane with Newtonian Heating (NH) condition. The problem is modeled in terms of partial differential equations with some physical conditions. Closed form solutions in terms of exponential and complementary error functions of Gauss are obtained by using the Laplace transform technique. They satisfy the governing equations and impose boundary and initial conditions. The present solution in the absence of microrotaion reduce to well-known solutions of Newtonian fluid. Graphs are plotted to study the effects of various physical parameters on velocity and microrotation. Numerical results for skin-friction and wall couple stress are computed in tables. Apart from the engineering point of view, the present article has strong advantage over the published literature as the exact solutions obtained here can be used as a benchmark for comparison with numerical/approximate solutions and experimental data.

Keywords: *Unsteady flow, Micropolar fluid, Wall couple stress, Newtonian heating, Closed form solutions*

1. Introduction

Newtonian fluids described by Navier-Stokes equations are limited in terms of their applications. It is because they cannot precisely describe the characteristics of several physiological fluids exhibit microscopic effects arising from the local structure and micro-motions of the fluid elements. Each particle of these fluids is shape dependent, may be shrink or expand. More exactly, they may rotate independently without rotation and movement of the fluid. This will require an additional equation corresponding to angular momentum. Eringen [1, 2] was the first who introduce the theory of micropolar fluids describe the microrotation and microinertia effects. The mathematical model of such type of fluids can be used to discuss the behavior of polymeric suspensions, dirty oils, animal blood with rigid cells, liquid crystals, lubricants and many other biological fluids [3, 4]. A substantial work has done on the micropolar fluid because of their important industrial applications. However, here we discuss some of the important studies most related to the present work. Amongst them, Agarwal *et al*. [5] have analysed the heat transfer flow of a micropolar fluid over a porous stationary wall. Kim [6] studied micropolar fluid flow with constant wall temperature which later on extended by Kim and Kim [7] by considering constant surface heat flux at the wall. Rahman and Sultana [8] examined radiation effect on micropolar fluid flow with variable heat flux. Reddy [9] considered the unsteady flow of a micropolar fluid with variable heat flux in the presence of magnetic field. In the same year, Pal and Talukdar [10] extended the same problem by considering chemical reaction effects. Amongst the various investigation on free convection flow of a micropolar fluid, the reader is referred to some new attempts made in [11-19], and the references therein.

In all of these studies cited above, the researchers usually use the constant or variable wall condition for temperature. However, there were several problems of physical interest where the heat is transported to the fluid via a bounding surface with a finite heat capacity and the above conditions fail to work and the Newtonian heating condition is incorporated. This idea was first introduced by Merkin [20] where he studied the boundary layer flow past a vertical plate and found the analytical asymptotic solution near the leading edge and the full solution along the whole plate is obtained numerically. Following the work of Merkin, Chaudhary and Jain [21], obtained the exact solution for boundary layer free convection flow of viscous fluid past a vertical plate with Newtonian heating. After that, various researchers studied the boundary layer heat transfer flows with Newtonian heating including the work of Mebine and Adigio [22], Narahari and Ishak [23] and Abid *et al*. [24, 25]. In all these studies, authors investigated the unsteady free convection flow of viscous fluid with Newtonian heating condition and exact solutions are obtained using the Laplace transform technique. In contrast, numerical solutions using using Runge-Kutta-Fehlberg with fourth-fifth order technique for the steady boundary layer heat transfer flow of micropolar fluid with Newtonian heating past a stretching surface are recently obtained by Qasim *et al*. [26]. The literature survey shows that most of the Newtonian heating problems are limited to the Newtonian fluid and mostly, they are solved using any numerical or approximate technique, see for example [27-31]. In fact the Newtonian heating problems even for viscous fluid when someone is interested to get the exact solutions are much complicated and limited to few problems only. For non-Newtonian fluids exact solutions of such problems are scarce.

On the other hand, the exact solutions of micropolar fluids are limited even to simple problems. Amongst the exact investigators, Sherief *et al*. [32] studied the unsteady flow of a micropolar over a suddenly moved horizontal plate using the Laplace transform technique. However, for inverse Laplace transform, they used complex inversion formula involving contour integration with several difficult and unsolved integrals. In the present problem, we extended this idea by taking into account the free convection phenomena. More exactly, our interest is to study the unsteady free convection flow of a micropolar fluid over an oscillating vertical plate instead of a suddenly moved plate together with Newtonian heating condition. The exact solutions are obtained by using the Laplace transform technique. However, for finding the inverse Laplace transform, we have used direct formulas instead of using contour integration and the resulting solutions do not involve complex integrations and even using numerical computations, and hence easy to analyse for exact solutions. The obtained solutions are written in simplified forms in terms of exponential and complementary error functions. The interesting feature of the present work is that these solutions do not involve complicated integrals which are mostly divergent and difficult to integrate even numerically as given by Sherief *et al*. [32]. On the other hand these solutions can be used as a check of accuracy for other micropolar fluid problems studied via approximate or numerical schemes.

2. Mathematical Formulation

Let us consider the unsteady boundary layer flow of an incompressible micropolar fluid in the region $y > 0$ driven by a plane surface located at $y = 0$ with a fixed end at $x = 0$. It is assumed that at the initial moment $t = 0$, both the plate and the fluid are at rest with constant temperature T_∞. At time $t = 0^+$ the plate begins to oscillate in its plane ($y = 0$) according to

$$\boldsymbol{V} = UH(t)\cos(\omega t)\mathbf{i};\ t > 0. \tag{1}$$

According to the Newtonian heating condition, the temperature of the fluid is proportional to local surface temperature T. Under the usual Boussinesq approximation, the simplified equations governing the flow [11, 32] are

$$\rho\frac{\partial u}{\partial t} = (\mu+\alpha)\frac{\partial^2 u}{\partial y^2} + \rho g\beta_f(T - T_\infty) + \alpha\frac{\partial N}{\partial y}, \tag{2}$$

$$\rho j\frac{\partial N}{\partial t} = \gamma_0\frac{\partial^2 N}{\partial y^2}, \tag{3}$$

$$\rho C_p\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial y^2}. \tag{4}$$

The appropriate initial and boundary conditions are given as:

$$u(y,0) = 0,\ N(y,0) = 0,\ T(y,0) = T_\infty,\ \text{for all } y \ge 0, \tag{5}$$

$$u(0,t) = H(t)U\cos(\omega t), N(0,t) = -n\frac{\partial u}{\partial y}(0,t),\ \frac{\partial T}{\partial y}(0,t) = -h_s T(0,t),\ t > 0, \tag{6}$$

$$u(\infty,t) \to 0,\ N(\infty,t) \to 0,\ T(\infty,t) \to T_\infty,\ t > 0. \tag{7}$$

To reduce the above equations into non-dimensional forms, we introduce the following non-dimensional quantities

$$y^* = \frac{U}{\nu}y,\ t^* = \frac{U^2}{\nu}t,\ u^* = \frac{u}{U},\ N^* = \frac{\nu}{U^2}N,\ \theta = \frac{T - T_\infty}{T_\infty},\ \omega^* = \frac{\nu}{U^2}\omega,\ j^* = \frac{U^2}{\nu^2}j. \tag{8}$$

Implementing equation (8) into equations (2)-(4), we obtain the following non-dimensional partial differential equations (*symbol is dropped for simplicity)

$$\frac{\partial u}{\partial t} = (1+\beta)\frac{\partial^2 u}{\partial y^2} + \beta\frac{\partial N}{\partial y} + Gr\theta, \tag{9}$$

$$\frac{\partial N}{\partial t} = \frac{1}{\eta}\frac{\partial^2 N}{\partial y^2}, \tag{10}$$

$$\Pr\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial y^2}. \tag{11}$$

The corresponding initial and boundary conditions in non-dimensional forms are

$$u(y,0)=0,\ N(y,0)=0,\ \theta(y,0)=0,\ \text{for all } y\ge 0, \tag{12}$$

$$u(0,t)=H(t)\cos(\omega t),\ N(0,t)=-n\frac{\partial u}{\partial y}(0,t),\ \frac{\partial\theta}{\partial y}(0,t)=-\gamma\left[1+\theta(0,t)\right],\ t>0, \tag{13}$$

$$u(\infty,t)\to 0,\ N(\infty,t)\to 0,\ \theta(\infty,t)\to 0,\ t>0, \tag{14}$$

where

$$Gr=\frac{\nu g\beta_f T_\infty}{U^3},\ \beta=\frac{\alpha}{\mu},\ \Pr=\frac{\mu C_p}{k},\ \eta=\frac{\mu j}{\gamma_0}=\left(\frac{2}{2+\beta}\right),\ \gamma=\frac{h_s\nu}{U}.$$

We note that equation (13) gives $\theta=0$ when $\gamma=0$, corresponding to have $h_s=0$ and hence no heat transfer from the plate exists [24, 27].

3. Analytical Solutions

Applying Laplace transforms to equations (9)-(11), and using conditions (12)-(14), we get the following solutions

$$\begin{aligned}
u(y,t)=&\left(\frac{1+a_3a_5}{4}\right)H(t)e^{-i\omega t}\left[e^{-y\sqrt{-i\omega a_8}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}-\sqrt{-i\omega t}\right)+e^{y\sqrt{-i\omega a_8}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}+\sqrt{-i\omega t}\right)\right]\\
&+\left(\frac{1+a_3a_5}{4}\right)H(t)e^{i\omega t}\left[e^{-y\sqrt{i\omega a_8}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}-\sqrt{i\omega t}\right)+e^{y\sqrt{i\omega a_8}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}+\sqrt{i\omega t}\right)\right]\\
&-\left(\frac{a_3a_5}{4}\right)H(t)e^{-i\omega t}\left[e^{-y\sqrt{-i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}-\sqrt{-i\omega t}\right)+e^{y\sqrt{-i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}+\sqrt{-i\omega t}\right)\right]\\
&-\left(\frac{a_3a_5}{4}\right)H(t)e^{i\omega t}\left[e^{-y\sqrt{i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}-\sqrt{i\omega t}\right)+e^{y\sqrt{i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}+\sqrt{i\omega t}\right)\right]\\
&+\frac{a_9}{a_1^2}\left[e^{\left(a_1^2t-ya_1\sqrt{a_8}\right)}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}-a_1\sqrt{t}\right)-\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}\right)\right]-\frac{a_9}{a_1}\left[2\sqrt{\frac{t}{\pi}}e^{-\frac{y^2a_8}{4t}}-y\sqrt{a_8}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}\right)\right]\\
&-a_9\left[\left(t+\frac{y^2a_8}{2}\right)\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}\right)-y\sqrt{\frac{a_8t}{\pi}}e^{-\frac{y^2a_8}{4t}}\right]+a_2\left[\left(t+\frac{y^2\Pr}{2}\right)\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}\right)-y\sqrt{\frac{\Pr t}{\pi}}e^{-\frac{y^2\Pr}{4t}}\right]\\
&-\frac{a_2}{a_1^2}\left[e^{\left(a_1^2t-ya_1\sqrt{\Pr}\right)}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}-a_1\sqrt{t}\right)-\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}\right)\right]+\frac{a_2}{a_1}\left[2\sqrt{\frac{t}{\pi}}e^{-\frac{y^2\Pr}{4t}}-y\sqrt{\Pr}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}\right)\right]\\
&+\frac{a_{10}}{a_1^2}\left[e^{\left(a_1^2t-ya_1\sqrt{\eta}\right)}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}-a_1\sqrt{t}\right)-\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right)\right]-\frac{a_{10}}{a_1}\left[2\sqrt{\frac{t}{\pi}}e^{-\frac{y^2\eta}{4t}}-y\sqrt{\eta}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right)\right]
\end{aligned}$$

$$-a_{10}\left[\left(t+\frac{y^2\eta}{2}\right)\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right)-y\sqrt{\eta}\sqrt{\frac{t}{\pi}}e^{-\frac{y^2\eta}{4t}}\right], \tag{15}$$

$$N(y,t)=\frac{a_5\omega}{4i\sqrt{-i\omega}}H(t)e^{-i\omega t}\left[e^{-y\sqrt{-i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}-\sqrt{-i\omega t}\right)-e^{y\sqrt{-i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}+\sqrt{-i\omega t}\right)\right]$$

$$-\frac{a_5\omega}{4i\sqrt{i\omega}}H(t)e^{i\omega t}\left[e^{-y\sqrt{i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}-\sqrt{i\omega t}\right)-e^{y\sqrt{i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}+\sqrt{i\omega t}\right)\right]-a_5\left[\frac{1}{\sqrt{\pi t}}e^{-\frac{y^2\eta}{4t}}\right]$$

$$-a_7\left[2\sqrt{\frac{t}{\pi}}e^{-\frac{y^2\eta}{4t}}-y\sqrt{\eta}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right)\right]+\frac{a_7}{a_1}\left[e^{\left(a_1^2t-ya_1\sqrt{\eta}\right)}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}-a_1\sqrt{t}\right)-\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right)\right], \tag{16}$$

$$\theta(y,t)=e^{\left(a_1^2t-ya_1\sqrt{\Pr}\right)}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}-a_1\sqrt{t}\right)-\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}\right), \tag{17}$$

where

$$a_1=\frac{\gamma}{\sqrt{\Pr}},\ a_2=\frac{Gr}{\Pr(1+\beta)-1},\ a_3=\frac{\beta\sqrt{\eta}}{\eta(1+\beta)-1},\ a_4=a_3\sqrt{\eta}-\frac{1}{n}-\frac{a_3}{\sqrt{1+\beta}},\ a_5=\frac{1}{a_4\sqrt{1+\beta}},$$

$$a_6=\frac{a_1a_2}{a_4}\sqrt{\Pr},\ a_7=\frac{1}{a_1}\left(a_6-a_1a_2a_5\right),\ a_8=\frac{1}{1+\beta},\ a_9=\frac{1}{a_1}\left(a_1a_2+a_1a_2a_3a_5-a_3a_6\right),$$

$$a_{10}=\frac{1}{a_1}\left(a_3a_6+a_1a_2a_3a_5\right),\ a_{11}=\frac{Gr}{\Pr-1}.$$

The above solution for velocity (15) and microrotation (16) are valid for $\Pr\neq a_8$. The corresponding solution for $\Pr=a_8$, can be easily obtained by substituting $\Pr=a_8$ into equation (11) and follow a similar procedure as discussed above. On the other hand, the expression of skin-friction for micropolar fluid is given as

$$\tau=(\mu+\alpha)\frac{\partial u}{\partial y}|_{y=0}+\alpha N|_{y=0}.$$

In non-dimensional form, we can write the above equation as

$$\tau^*=\frac{\tau}{\rho U^2}=(1+\beta)\frac{\partial u^*}{\partial y^*}|_{y^*=0}+\beta N^*|_{y^*=0}. \tag{18}$$

The expression of wall couple stress is given as

$$C_m=\gamma_0\frac{\partial N}{\partial y}|_{y=0},$$

and in non-dimensional form, we get

$$C_m^*=\frac{C_m}{\mu jU}=\frac{1}{\eta}\frac{\partial N^*}{\partial y^*}|_{y^*=0}. \tag{19}$$

For lack of space, the solutions for the skin-friction and wall couple stress are not considered, but are discussed numerically for different values of parameters in section 5.

4. Limiting Cases

This section includes some limiting cases of the present analytical solutions.

4.1 Stokes first problem

In this case we consider the flow situation when the fluid motion is induced by free convection together with sudden motion of the plate. In such a configuration the fluid is moving linearly and hence no oscillation is observed. In that case $\omega = 0$. In fluid mechanics such a physical configuration is known as Stokes first problem. These solutions can be obtained as a special case from equation (13), by taking $\omega = 0$, and is given as

$$\begin{aligned}
u(y,t) &= \operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}\right) + a_3 a_5\left[\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}\right)\right] - a_3 a_5\left[\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right)\right] \\
&+ \frac{a_9}{a_1^2}\left[e^{\left(a_1^2 t - y a_1\sqrt{a_8}\right)}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}} - a_1\sqrt{t}\right) - \operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}\right)\right] - \frac{a_9}{a_1}\left[2\sqrt{\frac{t}{\pi}}e^{-\frac{y^2 a_8}{4t}} - y\sqrt{a_8}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}\right)\right] \\
&- a_9\left[\left(t + \frac{y^2 a_8}{2}\right)\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}\right) - y\sqrt{\frac{a_8 t}{\pi}}e^{-\frac{y^2 a_8}{4t}}\right] + a_2\left[\left(t + \frac{y^2 \Pr}{2}\right)\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}\right) - y\sqrt{\frac{\Pr t}{\pi}}e^{-\frac{y^2 \Pr}{4t}}\right] \\
&+ \frac{a_2}{a_1}\left[2\sqrt{\frac{t}{\pi}}e^{-\frac{y^2 \Pr}{4t}} - y\sqrt{\Pr}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}\right)\right] - \frac{a_2}{a_1^2}\left[e^{\left(a_1^2 t - y a_1\sqrt{\Pr}\right)}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}} - a_1\sqrt{t}\right) - \operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}\right)\right] \\
&+ \frac{a_{10}}{a_1^2}\left[e^{\left(a_1^2 t - y a_1\sqrt{\eta}\right)}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}} - a_1\sqrt{t}\right) - \operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right)\right] - \frac{a_{10}}{a_1}\left[2\sqrt{\frac{t}{\pi}}e^{-\frac{y^2\eta}{4t}} - y\sqrt{\eta}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right)\right] \\
&- a_{10}\left[\left(t + \frac{y^2\eta}{2}\right)\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right) - y\sqrt{\frac{\eta t}{\pi}}e^{-\frac{y^2\eta}{4t}}\right],
\end{aligned} \tag{20}$$

$$\begin{aligned}
N(y,t) &= \frac{a_7}{a_1}\left[e^{\left(a_1^2 t - y a_1\sqrt{\eta}\right)}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}} - a_1\sqrt{t}\right) - \operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right)\right] \\
&- a_7\left[2\sqrt{\frac{t}{\pi}}e^{-\frac{y^2\eta}{4t}} - y\sqrt{\eta}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right)\right] - a_5\left[\frac{1}{\sqrt{\pi t}}e^{-\frac{y^2\eta}{4t}}\right].
\end{aligned} \tag{21}$$

Note that the solutions given by equations (20) and (21), are also new and not available in the published literature.

4.2 Absence of thermal effects

In the absence of free convection $(Gr = 0)$, the results for velocity and microrotation are obtained as:

$$u(y,t)=\left(\frac{1+a_3a_5}{4}\right)H(t)e^{-i\omega t}\left[e^{-y\sqrt{-i\omega a_8}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}-\sqrt{-i\omega t}\right)+e^{y\sqrt{-i\omega a_8}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}+\sqrt{-i\omega t}\right)\right]$$

$$+\left(\frac{1+a_3a_5}{4}\right)H(t)e^{i\omega t}\left[e^{-y\sqrt{i\omega a_8}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}-\sqrt{i\omega t}\right)+e^{y\sqrt{i\omega a_8}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{a_8}{t}}+\sqrt{i\omega t}\right)\right]$$

$$-\left(\frac{a_3a_5}{4}\right)H(t)e^{-i\omega t}\left[e^{-y\sqrt{-i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}-\sqrt{-i\omega t}\right)+e^{y\sqrt{-i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}+\sqrt{-i\omega t}\right)\right]$$

$$-\left(\frac{a_3a_5}{4}\right)H(t)e^{i\omega t}\left[e^{-y\sqrt{i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}-\sqrt{i\omega t}\right)+e^{y\sqrt{i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}+\sqrt{i\omega t}\right)\right], \tag{22}$$

$$N(y,t)=\frac{a_5\omega}{4i\sqrt{-i\omega}}H(t)e^{-i\omega t}\left[e^{-y\sqrt{-i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}-\sqrt{-i\omega t}\right)-e^{y\sqrt{-i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}+\sqrt{-i\omega t}\right)\right]$$

$$-\frac{a_5\omega}{4i\sqrt{i\omega}}H(t)e^{i\omega t}\left[e^{-y\sqrt{i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}-\sqrt{i\omega t}\right)-e^{y\sqrt{i\omega\eta}}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}+\sqrt{i\omega t}\right)\right]. \tag{23}$$

4.3 Newtonian fluid

In the absence of microrotation parameter $(\beta=0)$, the solution for velocity reduces to the corresponding solution for Newtonian fluid

$$u(y,t)=\frac{H(t)}{4}e^{-i\omega t}\left[e^{-y\sqrt{-i\omega}}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}-\sqrt{-i\omega t}\right)+e^{y\sqrt{-i\omega}}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{-i\omega t}\right)\right]$$

$$+\frac{H(t)}{4}e^{i\omega t}\left[e^{-y\sqrt{i\omega}}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}-\sqrt{i\omega t}\right)+e^{y\sqrt{i\omega}}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{i\omega t}\right)\right]$$

$$+\frac{a_{11}}{a_1}\left[2\sqrt{\frac{t}{\pi}}e^{-\frac{y^2\Pr}{4t}}-y\sqrt{\Pr}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}\right)-2\sqrt{\frac{t}{\pi}}e^{-\frac{y^2}{4t}}+y\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right]$$

$$+\frac{a_{11}}{a_1^2}\left[\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}\right)-e^{\left(a_1^2t-ya_1\sqrt{\Pr}\right)}\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}-a_1\sqrt{t}\right)\right]$$

$$-\frac{a_{11}}{a_1^2}\left[\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right)-e^{\left(a_1^2t-ya_1\right)}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}-a_1\sqrt{t}\right)\right]$$

$$+a_{11}\left[\left(\frac{y^2\Pr}{2}+t\right)\operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\Pr}{t}}\right)-y\sqrt{\Pr}\sqrt{\frac{t}{\pi}}e^{-\frac{y^2\Pr}{4t}}\right]$$

$$-a_{11}\left[\left(\frac{y^2}{2}+t\right)\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right)-y\sqrt{\frac{t}{\pi}}e^{-\frac{y^2}{4t}}\right]. \tag{24}$$

In addition, the above result is agrees with the result reported by Chaudhary and Jain [21], when $\omega\to0$ (impulsively motion of the plate) and $\gamma=1$.

5. Graphical Results and Discussion

In this work, an unsteady free convection flow of a micropolar fluid over a vertical plate oscillating in its own plane with Newtonian Heating (NH) condition is analysed. Closed form solutions are obtained by using the Laplace transform technique. In order to explore the physics of the problem, the obtained analytical results for velocity and microrotation are computed numerically and then plotted graphically for different flow parameters such as microrotation parameter β, dimensionless spin gradient viscosity parameter η, Prandtl number $\Pr$, Grashof number Gr, microelement n, conjugate parameter for Newtonian heating γ, time t and phase angle ωt. The numerical results for skin-friction and wall couple stress are computed and given in tables. The energy equation given by (4) is similar to equation (2) in [21], under the same boundary conditions. Therefore, the solution given by (17) is identical to equation (9) evaluated in [21]. Due to this reason, and in order to avoid the repetition, the corresponding graphical results are not included.

The velocity and microrotation profiles for different values of microrotation parameter β are shown in figs. 1(a) and 1(b). The graphs show that an increase in the microrotation parameter β results in increase in the velocity while decreases the microrotation. The influence of spin gradient viscosity parameter η on the velocity and microrotation is plotted in figs. 2(a) and 2(b). It is observed that velocity increases with increasing η, while reverse effect is observed for microrotation. Form fig. 3(a), it is noticed that the velocity decreases with increasing of Prandtl number. This situation is in consistence with the physical observation because fluids with large Prandtl number corresponds to the higher viscosity and smaller thermal conductivity, which makes the fluid thick and hence causes a decrease in velocity of fluid. The influence of the fluid motion near the plate is maximum and fades away as the distance from the plate increases and finally its velocity goes to zero as, $y \to \infty$. On the other hand, microrotation increases with the increasing values of $\Pr$ as shown in fig. 3(b). The influence of Grashof number Gr on velocity and microrotation profiles are shown in figs. 4(a) and 4(b). Here $Gr = 0$ represents the absence of free convection, while $Gr > 0$ corresponds the cooling problem. It is observed that velocity increases with the increasing values of Gr, while microrotation decreases with an increase in Gr. Further, from these figures, it is noticed that Grashof number does not have any influence as the fluid moves away from the bounding surface. Figs. 5(a) and 5(b) show the effects of microelement n, which relates to the microgyration vector and shear stress on velocity and microrotation profiles. It is observed that the velocity decreases with the increasing values of n, while microrotation increases with increase in n.

The physical behaviour of conjugate parameter γ, for velocity and microrotation profiles is studied in figs. 6(a) and 6(b). An increase in the conjugate parameter may reduce the fluid density and increase the momentum boundary layer thickness and finally increases the fluid motion. On the other hand, γ is found to have an opposite effect on microrotation profiles as observed for velocity. Figs. 7(a) and 7(b) demonstrate the effects of time t on the velocity and microrotation profiles. It is observed that velocity and microtation increase with an increase in t. The velocity and microtation profiles for different values of phase angle ωt are depicted in figs. 8(a) and 8(b). It is found that the velocity and microrotation show an oscillatory behavior. Three different values of phase angel are chosen. It is seen from fig. 8(a), that velocity profiles start from 1, 0 and -1 when the phase angel $0, \frac{\pi}{2}$ and π respectively. This graph satisfies the boundary condition (13), when we take $0, \frac{\pi}{2}$ and π. Similarly, the graphical results for microrotation are satisfying the imposed boundary conditions given by (13). Furthermore, the velocity and microrotation has maximum values near the plate and decreases with the increasing distance from the plate and approaches zero as $y \to \infty$. Finally, for the comparison of the present analysis with those existing in the literature, we have plotted fig. 9. It is found that for $\beta = \eta = 0$ and $n = 0.00001$ (equivalently $n \to 0$), our results are identical with those obtained in [24]. It is also found from this figure that in the presence microrotation parameter $(\beta \neq 0)$ as well as spin gradient viscosity parameter $(\eta \neq 0)$, the velocity has its maximum values. On the other hand, numerical results for the skin-friction and wall couple stress for different parameters are given in tables 1 and 2. The skin-friction is found to increase with increasing values of η, β, Gr, n, γ, t and ωt while it decreases with the increasing of $\Pr$. It is observed from tab. 2, that wall couple stress increases as Gr, n, γ, t and ωt are increased and decreases when η, β and $\Pr$ are increased.

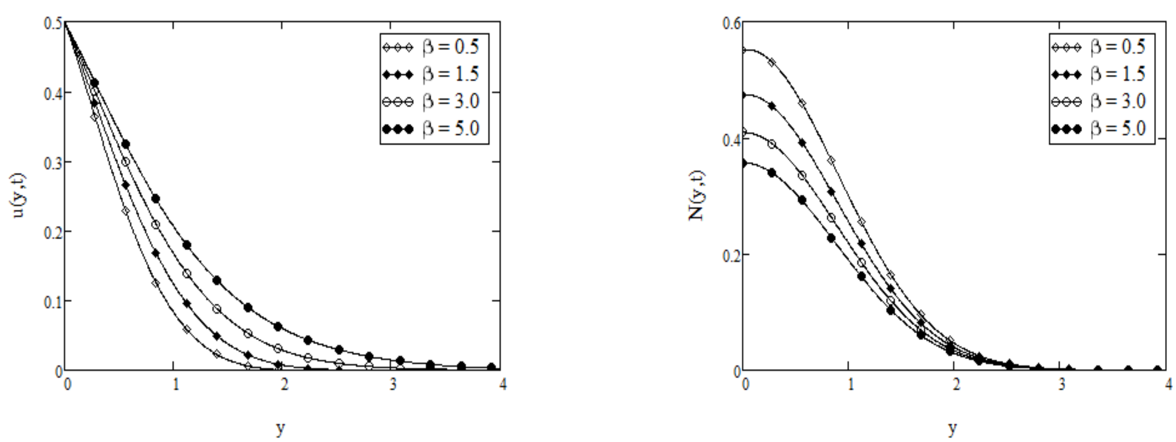

Fig. 1: Influence of β on (a) velocity and (b) microrotation, when $\eta = 0.5, \Pr = 3, Gr = 5, n = 0.5$, $\gamma = 0.5, t = 0.2, \omega t = \frac{\pi}{3}$.

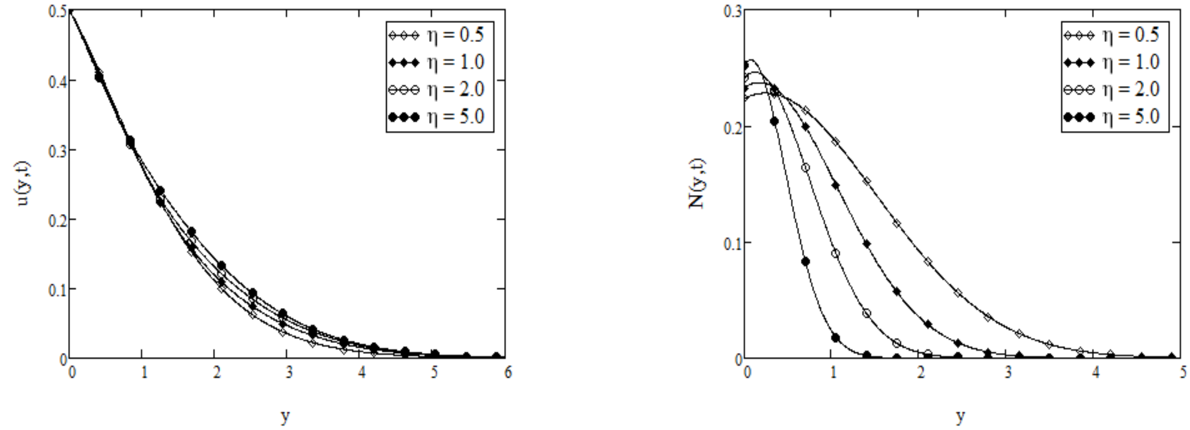

Fig. 2: Influence of η on (a) velocity and (b) microrotation, when $\beta = 3, \Pr = 3, Gr = 3, n = 0.5,$ $\gamma = 0.5, t = 0.5, \omega t = \dfrac{\pi}{3}.$

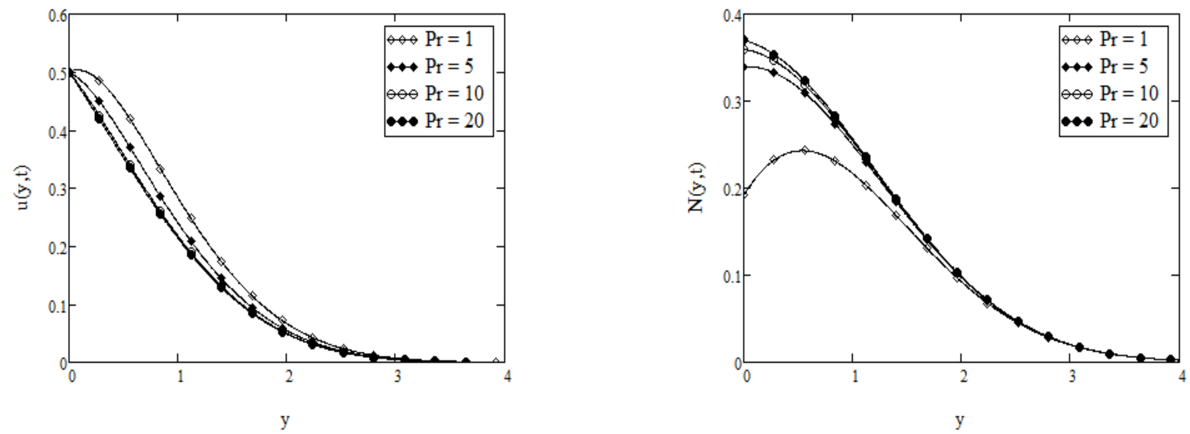

Fig. 3: Influence of $\Pr$ on (a) velocity and (b) microrotation, when $\eta = 0.5, \beta = 1.5, Gr = 5,$ $n = 0.5, \gamma = 0.5, t = 0.4, \omega t = \dfrac{\pi}{3}.$

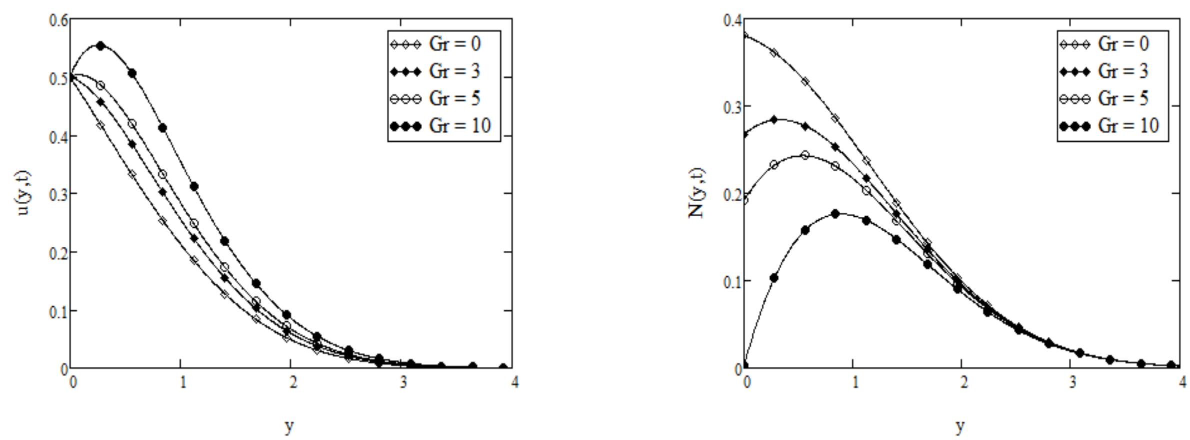

Fig. 4: Influence of Gr on (a) velocity and (b) microrotation, when $\eta = 0.5, \beta = 1.5, \Pr = 1,$ $n = 0.5, \gamma = 0.5, t = 0.4, \omega t = \dfrac{\pi}{3}.$

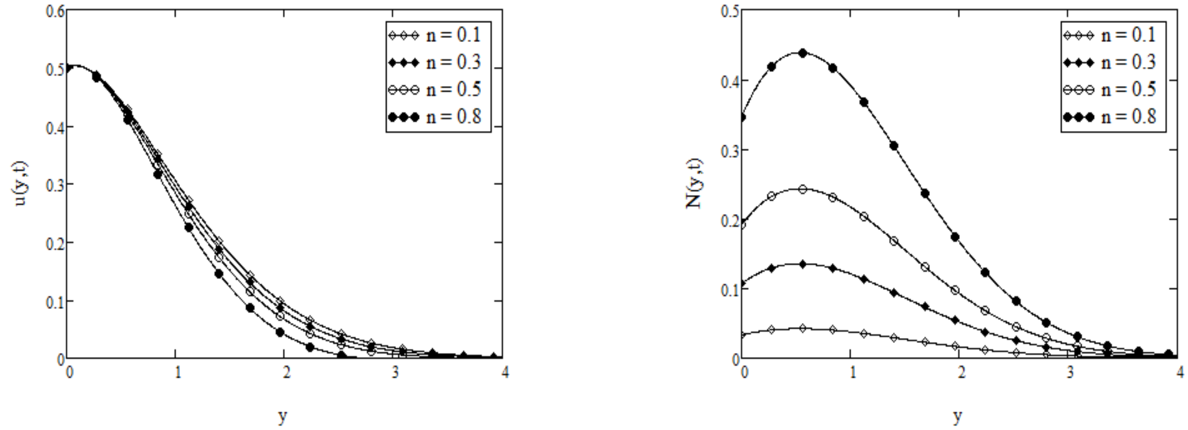

Fig. 5: Influence of n on (a) velocity and (b) microrotation, when $\eta = 0.5, \beta = 1.5, \Pr = 1, Gr = 5,$ $\gamma = 0.5, t = 0.4, \omega t = \frac{\pi}{3}.$

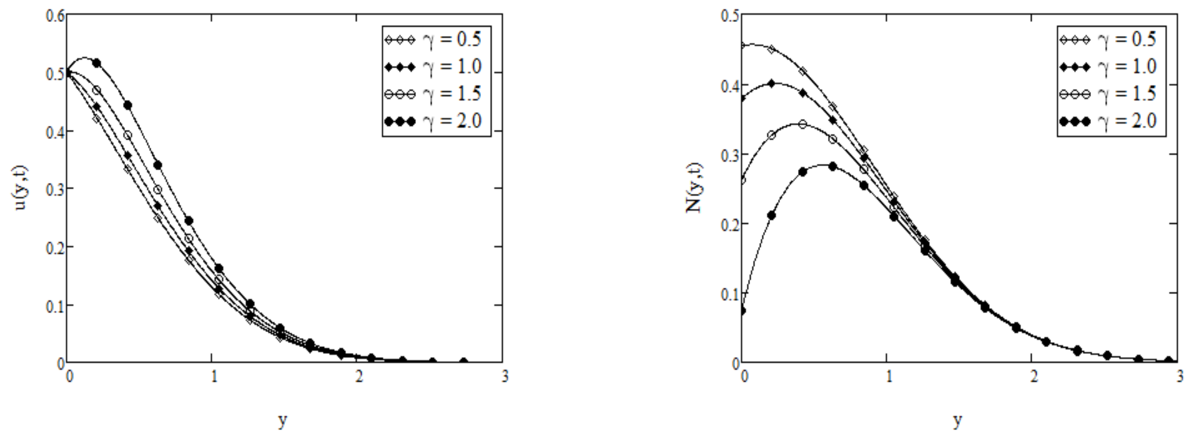

Fig. 6: Influence of γ on (a) velocity and (b) microrotation, when $\eta = 0.5, \beta = 1.5, \Pr = 1, Gr = 3,$ $n = 0.5, t = 0.2, \omega t = \frac{\pi}{3}.$

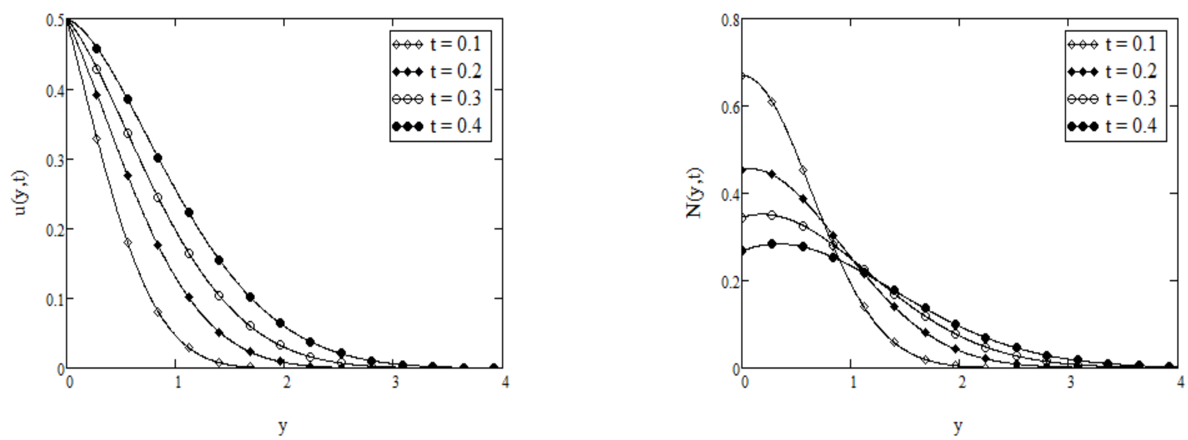

Fig. 7: Influence of t on (a) velocity and (b) microrotation, when $\eta = 0.5, \beta = 1.5, \Pr = 1, Gr = 3,$ $n = 0.5, \gamma = 0.5, \omega t = \frac{\pi}{3}.$

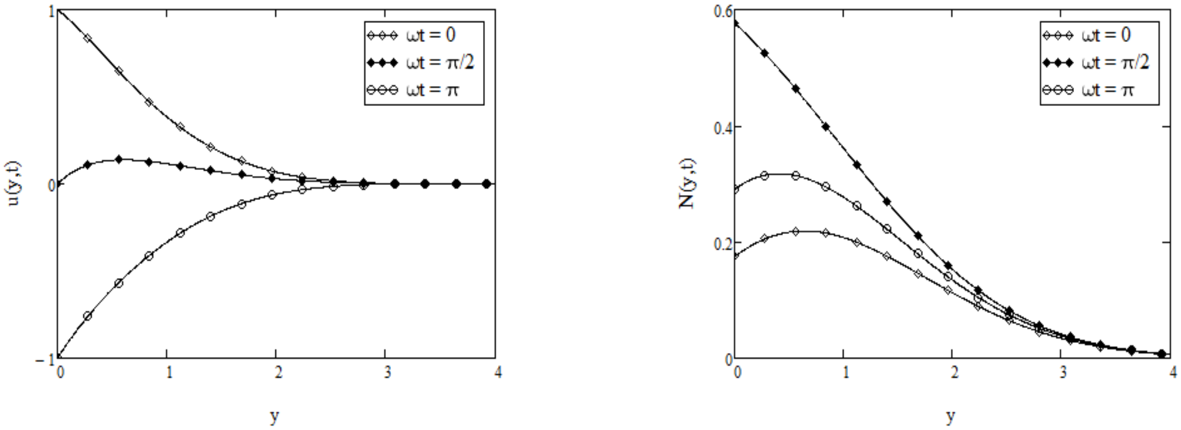

Fig. 8: Influence of ωt on (a) velocity and (b) microrotation, when $\eta = 0.5, \beta = 0.5, \Pr = 3,$ $Gr = 5, n = 0.5, \gamma = 0.5, t = 0.5.$

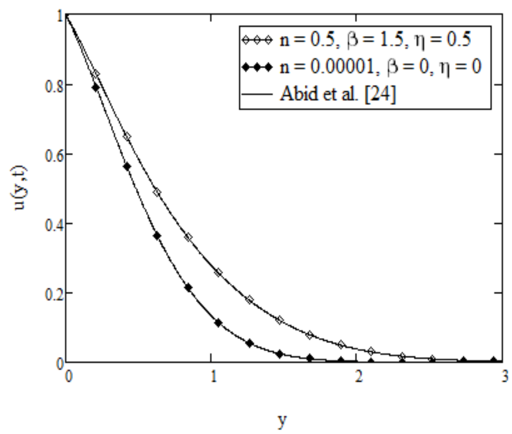

Fig. 9: Comparison of the present results at $\Pr = 0.7, Gr = 5, \gamma = 0.5,\ t = 0.2,\ \omega t = 0$ with those obtained in [24].

Table 1: Numerical results for skin friction

t	β	η	Pr	Gr	n	γ	ωt	τ^*
0.2	1.5	0.5	0.7	3	0.2	1	$\pi/2$	1.099
0.4	1.5	0.5	0.7	3	0.2	1	$\pi/2$	2.230
0.2	3.0	0.5	0.7	3	0.2	1	$\pi/2$	1.363
0.2	1.5	2.0	0.7	3	0.2	1	$\pi/2$	1.123
0.2	1.5	0.5	3.0	3	0.2	1	$\pi/2$	0.531
0.2	1.5	0.5	0.7	5	0.2	1	$\pi/2$	1.591
0.2	1.5	0.5	0.7	3	0.5	1	$\pi/2$	1.516
0.2	1.5	0.5	0.7	3	0.2	2	$\pi/2$	3.295
0.2	1.5	0.5	0.7	3	0.2	1	π	3.124

Table 2: Numerical results for wall couple stress

t	β	η	Pr	Gr	n	γ	ωt	C_m^*
0.2	1.5	0.5	0.7	3	0.2	1	$\pi/2$	0.224
0.4	1.5	0.5	0.7	3	0.2	1	$\pi/2$	0.464
0.2	3.0	0.5	0.7	3	0.2	1	$\pi/2$	0.154
0.2	1.5	2.0	0.7	3	0.2	1	$\pi/2$	0.114
0.2	1.5	0.5	3.0	3	0.2	1	$\pi/2$	0.019
0.2	1.5	0.5	0.7	5	0.2	1	$\pi/2$	0.399
0.2	1.5	0.5	0.7	3	0.5	1	$\pi/2$	0.623
0.2	1.5	0.5	0.7	3	0.2	2	$\pi/2$	1.161
0.2	1.5	0.5	0.7	3	0.2	1	π	0.262

6. Conclusions

The closed form solutions for unsteady free convection flow of a micropolar fluid with Newtonian heating are obtained. Laplace transform method is used for solution. These solutions are expressed in simpler forms in terms of exponential and complementary error functions. Graphs are plotted for embedded parameters and discussed. Numerical results for the skin-friction and wall couple stress are computed in tables. Results showed that velocity decreases significantly with the increasing of Prandtl number but it increases when either Grashof number or conjugate parameter is increased. The microrotation decreases with the increasing values of microrotation parameter. For correctness and verification, the present results are found in excellent agreement with published results for Newtonian fluid executing the same motion. Moreover, the classical solutions corresponding to the Stokes first problem for micropolar fluid are also obtained as a special case.

Nomenclature

C_p	Heat capacity at a constant pressure $\left[J\cdot kg^{-1}\cdot K^{-1}\right]$
Gr	Grashof number
h_s	Heat transfer coefficient
j	Microinertia per unit mass $\left[m^2\right]$
k	Thermal conductivity
n	Parameter related to microgyration vector and shear stress
N	Angular velocity $\left[m\cdot s^{-1}\right]$
Pr	Prandtl number
T	Fluid temperature $\left[K\right]$
T_∞	Ambient temperature $\left[K\right]$
t	Time $\left[s\right]$
U	Amplitude of plate oscillations $\left[m\right]$
u	Velocity of the fluid $\left[m\cdot s^{-1}\right]$
ωt	Phase angle
$H(t)$	Unit step function
erfc	Complementary error function

Greek symbols

α	Vortex viscosity $\left[kg\cdot m^{-1}\cdot s^{-1}\right]$
β	Microrotation parameter
β_f	Volumetric coefficient of thermal expansion $\left[K^{-1}\right]$
γ	Conjugate parameter

γ_0	Spin gradient viscosity $\left[kg \cdot m \cdot s^{-1}\right]$		ω	Frequency of oscillation
ρ	Fluid density $\left[kg \cdot m^{-3}\right]$		**Subscripts**	
μ	Dynamic viscosity $\left[kg \cdot m^{-1} \cdot s^{-1}\right]$		w	Condition at wall
η	Spin gradient viscosity parameter		∞	Condition at infinity
θ	Dimensionless temperature		**Superscripts**	
			$*$	Dimensional variables

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