UNSTEADY MAGNETOHYDRODYNAMICS THIN FILM FLOW OF A THIRD GRADE FLUID OVER AN OSCILLATING INCLINED BELT EMBEDDED IN A POROUS MEDIUM

by

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In the present work we examine the motion of an incompressible unidirectional magnetohydrodynamics thin film flow of a third grade fluid over an oscillating inclined belt embedded in a porous medium. Moreover, heat transfer analysis has been also discussed in the present work. This physical problem is modeled in terms of non-linear partial differential equations. These equations together with physical boundary conditions are solved using two analytical techniques namely optimal homotopy asymptotic method and homotopy perturbation method. The comparisons of these two methods for different time level are analyzed numerically and graphically. The results exposed that both methods are in closed agreement and they have identical solutions. The effects of various non-dimensional parameters have also been studied graphically.

Key words: unsteady thin film flows, magnetohydrodynamics, porous medium, third grade fluid, heat transfer, inclined belt, optimal homotopy asymptotic method, homotopy analysis method

Introduction

Non-Newtonian fluid flow and heat transfer acting a vital position in numerous technological and industrials manufacturing processes. A number of applications of non-Newtonian flow are found in drilling mud, polymer solution, drilling of gas and oil wells, glass fiber, and paper production. Third grade fluid is one of the significant sub-classes of non-Newtonian fluid. In literature the study of third grade fluid flow through various geometrical planes has received enormous concentration from scientists and engineers. Gul et al. [1] investigated unsteady second grade thin film fluid on a vertical belt. They studied thin film fluid motion at different time level. Aiyesimi et al. [2, 3] studied the cause of slip boundary on MHD third order fluid through inclined plane in the presence of heat transfer. They solved the problem by using regular and homotopy perturbation methods (HPM) and discussed the effect of physical parameters graphically. Abdullah [4] used HAM for the solution of non-linear problems. Gul et al. [5, 6] studied thin film flow of third order fluid in lifting and drainage problems for constant and variable viscosities. For solutions they used two analytical methods adomian decomposition method (ADM) and optimal homotopy asymptotic method (OHAM) for lifting and drainage velocity and temperature profiles. In their work, they also

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examined the comparison of these two solutions graphically and numerically. Some other interesting non-linear problems related to the present work are studied in [7]. Hayat et al. [8] discussed second grade fluid with convection boundary conditions. They used transformation to reduce partial differential equations to ordinary differential equations.

The flow through porous medium has countless applications in science and engineering such as petroleum, chemical engineering, chemical reactor, irrigation, and drainage. Hayat et al. [9] discussed the unsteady flow of third grade in a porous space. In the modeling of fluid flow they used the modified Darcy’s law. Gamal [10] studied the thin film flow of unsteady micro polar fluid through porous medium in the presence of MHD. They have been used numerical techniques to solve the problem. In their work the effects of the various modeled parameters have been presented graphically. Sajid et al. [11] discussed the thin film flow of fourth order fluid through a vertical cylinder.

The main objective if this works to study the unsteady MHD thin film of non-Newtonian fluid with heat transfer on an inclined oscillating belt using OHAM and HPM. Idrees et al. [12] discussed the axisymmetric flow of incompressible fluid between two parallel plates and analytic solutions are obtained using OHAM, HPM, and perturbation method. It was shown that OHAM solutions are more precise and accurate. Siddiqui et al. [13] studied the thin film flow of non-Newtonian fluid on inclined plane. The problem has been solved for velocity filed by using OHAM and perturbation technique. Mabood et al. [14] discussed the OHAM method for non-linear Riccati differential equation. Ganji and Rafei [15] investigated the HPM method for the solution of Hirota Stuitsuma coupled partial differential equations. Lin [16] studied the solution of partial differential equation using HPM. Nawaz et al. [17] studied the approximate solution of Burger’s equations using OHAM and compared the solution with ADM. Hemeda [18] discussed the consistent behavior of HPM for frictional order linear and non-linear partial differential equations. Moreover, the related work with this article can be seen in [19-23].

**Basic equation**

The MHD flow of incompressible fluid is based on the Darcy’s law, continuity, momentum and heat equations for third grade fluid given by:

\[
\begin{align*}
\mathbf{r} &= -\left(\mu + \alpha \frac{\partial}{\partial t}\right) \frac{\partial \mathbf{v}}{\partial x} \\
\nabla \cdot \mathbf{v} &= 0 \\
\rho \frac{D\mathbf{v}}{Dt} &= -\nabla p + \text{div}\tau + \rho g \sin \theta + \mathbf{j} \times \mathbf{B} + \mathbf{r} \\
\rho c_p \frac{D\theta}{Dt} &= k \nabla^2 \theta + \text{tr}(\mathbf{T} \cdot \mathbf{L})
\end{align*}
\]

where \(D/Dt\) is the material time derivative, \(\rho\) – the fluid density, \(\mathbf{v}\) is – the velocity vector of the fluid, \(g \sin \theta\) – the external body force, \(\mathbf{j} \times \mathbf{B} = [0, \sigma B_t^2 v(y), 0]\) – the Lorentz force per unit volume, \(\mathbf{J} = \sigma(\mathbf{F} + v \times \mathbf{B})\) is the current density, \(\mathbf{B} = (0, B_y, 0)\) – the uniform magnetic field, \(\sigma\) – the electrical conductivity, \(\mu\) – the dynamic viscosity, \(\theta\) – the temperature, \(k\) – the thermal conductivity, \(c_p\) – the specific heat, \(\phi\) – the porosity, \(K_1\) – the Darcy permeability, and \(\mathbf{T}\) – the Caushy stress tensor is define for third grade fluid as:
\( T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_2^2 + \beta_1 A_3 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (tr A_2^2) A_1 \)  

(5)

\[ A_1 = L + L', \quad L = \text{grad} \nu \]

(6)

\[ A_n = \frac{\partial A_{n-1}}{\partial t} + A_{n-1} L + L^T A_{n-1}, \quad n \geq 1 \]

(7)

where \( pI \) is the isotropic stress, \( A_1 \) and \( A_2 \) are the Rivlin Ericksen stress tensor, \( \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 \) are the material constant. For third grade fluid:

\[ \mu \geq 0, \quad \alpha_1 \geq 0, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0, \quad |\alpha_1 + \alpha_2| \leq 24\mu\beta_3 \]

Equations (1-8) are related with modified Darcy's law mentioned in [9, 10]

Formulation of problem

Consider a wide porous inclined belt. We consider a thin layer of third order fluid of uniform thickness \( \delta \) draining down the belt. The belt is oscillating and the fluid drains down the belt due to gravity. Uniform magnetic field is applied to the belt transversely. The \( x \)-axis is taken parallel to the belt and \( y \)-axis is perpendicular to the belt. Assuming that the flow is laminar and unsteady, ambient air pressure is absent whereas the fluid shear forces keep gravity balanced and the thickness of the film remains constant.

The velocity and temperature field yields the form:

\[ V = [\nu(y, t), 0, 0], \quad \text{and} \quad \theta = \theta(y, t) \]

(8)

The flow is under the following oscillating boundary conditions:

\[ \nu(0, t) = V \cos \omega t, \quad \frac{\partial \nu(\delta, t)}{\partial y} = 0 \]

(9)

\[ \theta(0, t) = \theta_0, \quad \theta(\delta, t) = \theta_t \]

(10)

where \( \omega \) is the frequency of the oscillating belt and \( \theta \) denote temperature.

According to the previous assumptions the momentum and energy eqs. (3) and (4) yield the form:

\[ \rho \frac{\partial \nu}{\partial t} = \frac{\partial}{\partial y} T_{xy} + \rho g \sin \theta - \sigma B_0^2 \nu + r \]

(11)

\[ \rho c_p \frac{\partial \theta}{\partial t} = k \left( \frac{\partial^2 \nu}{\partial y^2} \right) + T_{xy} \left( \frac{\partial \nu}{\partial y} \right) \]

(12)

The Cauchy stress component, \( T_{xy} \), of the third order fluid is:

\[ T_{xy} = \mu \frac{\partial \nu}{\partial y} + \alpha_1 \frac{\partial}{\partial t} \left( \frac{\partial \nu}{\partial y} \right) + \beta_1 \frac{\partial^2}{\partial t^2} \left( \frac{\partial \nu}{\partial y} \right) + 2(\beta_2 + \beta_3) \left( \frac{\partial \nu}{\partial y} \right)^3 \]

(13)

Inserting eq. (13) in to eqs. (11) and (12) the momentum and energy equations are reduced:

\[ \rho \frac{\partial \nu}{\partial t} = \mu \frac{\partial^2 \nu}{\partial y^2} + \rho \alpha_1 \frac{\partial}{\partial t} \left( \frac{\partial^2 \nu}{\partial y^2} \right) + 6 \beta_3 \left( \frac{\partial \nu}{\partial y} \right)^2 \frac{\partial^2 \nu}{\partial y^2} - \rho g \sin \theta - \sigma B_0 \nu + r \]

(14)

\[ \rho c_p \left( \frac{\partial \theta}{\partial t} \right) = k \left( \frac{\partial^2 \theta}{\partial y^2} \right) + \left( \frac{\partial \nu}{\partial y} \right) + \alpha_1 \frac{\partial \nu}{\partial y} \frac{\partial ^2 \nu}{\partial y^2} + 2 \beta_3 \left( \frac{\partial \nu}{\partial y} \right)^4 \]

(15)
Introducing non-dimensional variables:

\[ \bar{v} = \frac{v}{v}, \quad \bar{y} = \frac{y}{\delta}, \quad \bar{t} = \frac{t \mu}{\delta^2 \rho}, \quad \alpha = \frac{\alpha_1}{\delta^2 \rho}, \quad \beta = \frac{B_s v^2}{\mu \delta^2}, \quad S_t = \frac{\rho \delta^2 \sigma \delta \sin \theta}{\mu v}, \quad M = \frac{\delta^2 \sigma}{\kappa^2}; \]

\[ B_r = \frac{v^2 \mu}{k(\theta_1 - \theta_0)}, \quad P_r = \frac{c_p \mu}{k}, \quad \bar{\sigma} = \frac{\theta - \theta_0}{\theta_1 - \theta_0}, \quad Y_1 = \frac{1}{1 + \alpha \phi_1}, \quad Y_2 = \frac{\alpha}{1 + \alpha \phi_1}, \quad Y_3 = \frac{6 \beta}{1 + \alpha \phi_1}, \quad Y_4 = \frac{\phi_1}{1 + \alpha \phi_1}, \quad \phi_5 = \frac{2 \beta \phi_1}{1 + \alpha \phi_1}, \quad \phi_6 = \frac{S_2}{1 + \alpha \phi_1}. \quad (16) \]

where \( M \) is the magnetic parameter, \( \beta \) – the non-Newtonian parameter, \( S_t \) – the stock number, and \( \alpha, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 \) are the non-dimensional variables.

Inserting eq. (16) into eqs. (14) and (15), we obtain:

\[ \frac{\partial n}{\partial t} = Y_1 \frac{\partial^2 v}{\partial y^2} + Y_2 \frac{\partial}{\partial t} \left( \frac{\partial^2 v}{\partial y^2} \right) + Y_3 \left( \frac{\partial^2 v}{\partial t \partial y} \right)^2 - Y_4 v - Y_5 v \left( \frac{\partial^2 v}{\partial y^2} \right)^2 - \gamma_6 \quad (17) \]

\[ \frac{\partial \theta}{\partial t} = \frac{1}{r} \frac{\partial}{\partial y} \left( \frac{\partial^2 \theta}{\partial y^2} \right) + \frac{\partial v}{\partial y} \left( \frac{\partial^2 \theta}{\partial y^2} \right) + \alpha \left( \frac{\partial^2 \theta}{\partial t \partial y} \right) + 2 \beta \left( \frac{\partial^2 \theta}{\partial y^2} \right)^4 \quad (18) \]

The appropriate oscillating boundary conditions are reduced:

\[ \nu(0, t) = \cos \omega t, \quad \frac{\partial \nu(t)}{\partial y} = 0 \quad (19) \]

\[ \theta(0, t) = 0, \quad \theta(1, t) = 1 \quad (20) \]

The **OHAM solution**

In this section, we apply OHAM method on eqs. (17) and (18) together with boundary condition (19) and (20) and study zero, first and second component problems.

Zero and first component problems of velocity and temperature profiles are:

\[ p^0: \quad \frac{\partial^2 \nu_0(y, t)}{\partial y^2} = \frac{\gamma_6}{\gamma_1} \quad (21) \]

\[ \frac{\partial^2 \theta_0(y, t)}{\partial y^2} = 0 \quad (22) \]

\[ p^1: \quad \frac{\partial^2 \nu_1(y, t)}{\partial y^2} = -c_1 \frac{\partial \nu_0}{\partial t} - \gamma_4 c_1 v_0 + \gamma_6 + c_1 y_6 + c_1 v_0 y_5 \left( \frac{\partial \nu_0}{\partial y} \right)^2 + \gamma_1 \frac{\partial^2 v_0}{\partial y^2} (1 + c_1) + c_1 \gamma_3 \left( \frac{\partial^2 \nu_0}{\partial y^2} \right)^2 + c_1 \gamma_2 \frac{\partial}{\partial t} \left( \frac{\partial^2 \nu_0}{\partial y^2} \right) \quad (23) \]

\[ \frac{\partial^2 \theta_0}{\partial y^2} = -P_r \gamma_3 \frac{\partial \theta_0}{\partial t} + B_r \gamma_3 \left[ \left( \frac{\partial \theta_0}{\partial y} \right)^2 + 2 \beta \left( \frac{\partial \theta_0}{\partial y} \right)^4 + \alpha \frac{\partial \theta_0}{\partial y} \frac{\partial}{\partial t} \left( \frac{\partial \theta_0}{\partial y} \right) + \frac{\partial^2 \theta_0}{\partial y^2} (1 + c_3) \right] \quad (24) \]

Solutions of eqs. (21- 24) using boundary conditions (19, 20) are:

\[ \nu_0(y, t) = \cos[\omega t] - \left[ \cos[\omega t] - \frac{\gamma_6}{\gamma_1} \right] y - \left[ \frac{\gamma_6}{\gamma_1} \right] y^2 \quad (25) \]
\[ \theta_0(y, t) = y \] 

(26)

\[ v_{11} = \frac{(4\gamma_3 - \gamma_2)\cos(t\omega)^2\gamma_6}{8\gamma_1^2} - \frac{\omega\sin(t\omega)}{3\gamma_1} + \frac{(y_3 + \gamma_2\cos(t\omega)^2\gamma_6)^2\cos(t\omega)}{3\gamma_1} + \frac{y_3\gamma_6}{24\gamma_1^2} + \frac{(2\gamma_3 + 3\gamma_2)^2\cos(t\omega)\gamma_6}{120\gamma_1^3} + \frac{(2\gamma_3 + \gamma_2)^2\gamma_6^2}{480\gamma_1^4} \] 

(27)

\[ v_{12} = \frac{\omega\sin(t\omega)}{2\gamma_1} - \frac{(y_3 + \gamma_2\gamma_6\cos(t\omega)^2\gamma_6)^2\cos(t\omega)}{2\gamma_1} + \frac{(y_3 + \gamma_2\gamma_6\cos(t\omega)^2\gamma_6)^2\cos(t\omega)}{2\gamma_1} + \frac{(4\gamma_3 - \gamma_2)^2\gamma_6^2\cos(t\omega)}{2\gamma_1} + \frac{y_3\gamma_6^2}{8\gamma_1^2} \] 

(28)

\[ v_{13} = \frac{(2\gamma_3 + 2\gamma_2\gamma_6\cos(t\omega)^2 - \gamma_2\gamma_6\cos(t\omega))\cos(t\omega)}{12\gamma_1^2} - \frac{\omega\sin(t\omega)}{6\gamma_1} - \frac{(8\gamma_3 + 7\gamma_2\gamma_6\cos(t\omega)^2\gamma_6)^2}{24\gamma_1^2} + \frac{y_3\gamma_6}{12\gamma_1^2} + \frac{(8\gamma_3 - \gamma_2)^2\gamma_6^2}{480\gamma_1^4} \] 

(29)

\[ v_{14} = \frac{y_3\gamma_6^2}{24\gamma_1^2} + \frac{\cos(t\omega)^2\gamma_6}{24\gamma_1^2} \left(5\cos(t\omega) - 7\right) - \frac{y_3^2}{12\gamma_1^2} \left(y_3 + 5\gamma_6\right) \] 

(30)

\[ v_{15} = \frac{\cos(t\omega)\gamma_6^2 y_3^2}{10\gamma_1^2} - \frac{y_3\gamma_6^2}{20\gamma_1^2} \] 

(31)

\[ v_{16} = \frac{y_3\gamma_6^2}{60\gamma_1^4} \] 

(32)

\[ v_1(y, t) = c_1[v_{11}y + v_{12}y^2 + v_{13}y^3 + v_{14}y^4 + v_{15}y^5 + v_{16}y^6] \] 

(33)

\[ \theta_{11} = \frac{\cos(t\omega)(2\omega\sin(t\omega) - \beta\cos(3t\omega)) - (2 + 3\beta)\cos(t\omega)^2\gamma_6^2 + (1 + 3\beta)\gamma_6\cos(3t\omega) - \frac{\omega\sin(t\omega)\gamma_6}{12\gamma_1^2} - \frac{(1 + 6\beta)\gamma_6^2}{4\gamma_1^2} + \frac{\beta\cos(2t\omega)\gamma_6^2}{10\gamma_1^2} - \frac{c_3\gamma_6^2}{80\gamma_1^4}}{6\gamma_1} \] 

(34)

\[ \theta_{12} = \frac{(2 + 3\beta)\cos(t\omega)\gamma_6}{2} + \frac{1}{4} \cos(t\omega)(\beta\cos(3t\omega) - 2\omega\sin(t\omega)) - \frac{(1 + 3\beta)\gamma_6\cos(t\omega) - \beta\cos(3t\omega)\gamma_6}{2\gamma_1} + \frac{\omega\sin(t\omega)\gamma_6}{6\gamma_1} + \frac{(1 + 6\beta)\gamma_6^2}{8\gamma_1^2} + \frac{3\beta\cos(2t\omega)\gamma_6^2}{4\gamma_1^2} - \frac{\beta\cos(t\omega)\gamma_6^2}{2\gamma_1} \] 

(35)

\[ \theta_{13} = \frac{(1 + 3\beta)\gamma_6\cos(t\omega)\gamma_6}{3\gamma_1} + \frac{\beta\cos(3t\omega)\gamma_6}{3\gamma_1} - \frac{\omega\sin(t\omega)\gamma_6}{6\gamma_1} - \frac{(1 + 3\beta)\gamma_6^2}{6\gamma_1^2} - \frac{\beta\cos(2t\omega)\gamma_6^2}{2\gamma_1^2} + \frac{\beta\cos(t\omega)\gamma_6^2}{2\gamma_1^2} - \frac{y_3^2}{6\gamma_1^2} \] 

(36)

\[ \theta_{14} = \frac{(1 + 6\beta)\gamma_6^2}{12\gamma_1^2} + \frac{\beta\cos(2t\omega)\gamma_6^2}{2\gamma_1^2} - \frac{\beta\cos(t\omega)\gamma_6^2}{2\gamma_1^2} + \frac{y_3^2}{4\gamma_1^2} \] 

(37)

\[ \theta_{15} = \frac{2\beta\cos(t\omega)\gamma_6^2}{5\gamma_1^2} - \frac{y_3^2}{5\gamma_1^2} \] 

(38)

\[ \theta_{16} = \frac{y_3^2}{15\gamma_1^4} \] 

(39)

\[ \theta_1(y, t) = B_r c_3[\theta_{11}y + \theta_{12}y^2 + \theta_{13}y^3 + \theta_{14}y^4 + \theta_{15}y^5 + \theta_{16}y^6] \] 

(40)
The solutions of second component of velocity and temperature fields are too large. Therefore, the numerical solutions are given up to first order while, graphical solutions are given up to second order.

The value of \( c_1 \) for the velocity components are \( c_1 = -1.32285737 \), and \( c_2 = -1.15543843 \).

Also the values of \( c_\ell \) for the temperature distribution are:

\[ c_1 = 0.02431759, \quad c_2 = -0.10984471, \quad c_3 = -1.88757428, \quad c_4 = -0.59500373 \]

**The HPM solution**

In this section we apply HPM method on eqs. (17) and (18) together with boundary conditions (19, 20) and study zero, first and second component problems.

Zero and first order problems are:

\[
p^0: \quad \frac{\partial^2 \nu_0(y,t)}{\partial y^2} + \gamma_6 = 0 \tag{41}
\]

\[
\frac{\partial^2 \theta_0(y,t)}{\partial y^2} = 0 \tag{42}
\]

\[
p^1: \quad \frac{\partial^2 \nu_1(y,t)}{\partial y^2} y_1 = -\frac{\partial \nu_0}{\partial t} - \nu_0 y_4 + 2\gamma_6 - \nu_0 y_5 \left( \frac{\partial \nu_0}{\partial y} \right)^2 + 2\gamma_4 \left( \frac{\partial^2 \nu_0}{\partial y^2} \right) + \gamma_3 \left( \frac{\partial^2 \theta_0}{\partial y^2} \right) + \frac{1}{\gamma_2} \frac{\partial^2 \theta_0}{\partial y^2} \tag{43}
\]

Using boundary conditions (19) and (20) into eqs. (41)-(44), the component solutions are:

\[
\nu_0(y,t) = \cos[t\omega] - \left[ \cos[t\omega] - \frac{y_6}{2\gamma_1} \right] y - \left[ \frac{y_6}{2\gamma_1} \right] y^2 \tag{45}
\]

\[
\theta_0(y,t) = y \tag{46}
\]

\[
\nu_{11} = \left( \frac{4y_4 - y_6}{y_4} \right) y_4 \cos[t\omega] - \frac{\omega \sin[t\omega]}{3\gamma_1} + \left( \frac{y_4 + y_6 \cos[t\omega]}{2\gamma_1} \right) \cos[t\omega] + \frac{y_6 y_2}{24\gamma_1} + \frac{(20y_4 + 3y_2)y_4 \cos[t\omega]}{120y_1^3} + \frac{2(2y_4 + y_6)y_2 \cos[t\omega]}{480y_1^4} \tag{47}
\]

\[
\nu_{12} = \frac{\omega \sin[t\omega]}{2\gamma_1} - \left( \frac{y_4 + y_6 \cos[t\omega]}{2\gamma_1} \right) \cos[t\omega] - \frac{y_4 y_6}{8\gamma_1^3} + \frac{(4y_4 - y_6) y_4 \cos[t\omega]}{8\gamma_1^3} - \frac{y_6 y_2}{8\gamma_1^3} \tag{48}
\]

\[
\nu_{13} = \frac{\omega \sin[t\omega]}{2y_1} - \left( \frac{y_4 + y_6 \cos[t\omega]}{2\gamma_1} \right) \sin[t\omega] - \frac{7\cos[t\omega] y_4 y_6}{12\gamma_1^2} - \frac{(8y_4 + 7y_2)y_4 \cos[t\omega]}{24\gamma_1^3} - \frac{y_6 y_2}{12\gamma_1^2} - \frac{(y_4 - y_6) y_2}{48\gamma_1^4} \tag{49}
\]

\[
\nu_{14} = \frac{8\cos[t\omega] y_4 y_6}{24\gamma_1^2} - \frac{7\cos[t\omega] y_4 y_2}{24\gamma_1^3} + \frac{4y_6 y_2}{24\gamma_1^2} - \frac{y_2}{12\gamma_1^2} (8y_3 + 5y_5) \tag{50}
\]
The solution of second component of velocity and temperature distribution is too large. Therefore, the expressions of solutions are given up to first order while graphical solutions are given up to second order.

**Results and discussion**

In this paper, we examined the approximate analytical solutions for both velocity and temperature distribution of unsteady MHD thin film flow of non-Newtonian fluid through porous and oscillating inclined belt. The arising non-linear partial differential equations are solved using OHAM and HPM methods. The results of both methods are compared numerically and graphically for velocity and temperature distribution. The results obtained from OHAM and HPM are in excellent agreement. In tabs. 1 and 2 we investigated the numerical comparisons of these methods along with absolute error at different time level. Figure 1 shows...
The effects of various non-dimensional physical parameters are discussed in Figs. 8 to 9 for velocity and temperature distribution. Figures 8 and 9 show the variation in $\gamma_1$ on velocity and temperature distribution. It is revealed that velocity and temperature fields decrease by increasing $\gamma_1$. Moreover, the influence of the model parameters $\gamma_2$ and $\gamma_3$ are indicated in Figs. (10)-(13), respectively. These figures illustrate that $\gamma_2$ and $\gamma_3$ have opposite roles on the geometry of the problem. Figures 2 and 3 shows the graphical comparison of OHAM and HPM solutions at different values of physical parameters. Figures 4 and 5 give the influence of different time level on velocity and temperature distribution. Figures 6 and 7 show the velocity and temperature distribution by taking the different values of the physical domain oscillates with the belt oscillation. Increasing the non-Newtonian parameter, $\beta$, of the third order fluid causes more thickening of the boundary layer. In general, for the case with suction through the porous belt the Newtonian thin layer is thinner than that for the third order fluid layer. For a blowing through the porous belt, the fluids behave opposite to the case of suction. In case of Increasing, if the blowing velocity increases for the Newtonian fluid, then the shear boundary layer of the Newtonian fluid layer becomes thicker quickly, whilst for the third order fluids the thickness of boundary layers is not so thinner to variations in the blowing and only small thinning of the boundary layer happens. The effects of various non-dimensional physical parameters are discussed in Figs. 8 to 9 for velocity and temperature distribution. Figures 8 and 9 show the variation in $\gamma_1$ on velocity and temperature distribution. It is revealed that velocity and temperature fields decrease by increasing $\gamma_1$. Moreover, the influence of the model parameters $\gamma_2$ and $\gamma_3$ are indicated in Figs. (10)-(13), respectively. These figures illustrate that $\gamma_2$ and $\gamma_3$ have opposite roles on the
velocity and temperature fields. These figures show that velocity field increases for large values of $\gamma_2$ whereas temperature field is unchanged and velocity decreases for increasing $\gamma_2$.

It is clear from these figures that for $\gamma_2 < 1$ and $\gamma_3 < 1$, the fluids tend to Newtonian status. Further, it is clear from figs. (14)-(19) that due to the friction of no-slip boundary fluid along with the belt renders oscillation in the same stage and the amplitude. The amplitude of the velocity decreases rapidly towards the free surface of the inclined belt.

Figures 2 and 3. Comparison of OHAM and HPM methods for velocity profile (on the left) by taking $\omega = 0.2$, $\gamma_1 = 0.1$, $\gamma_2 = 0.2$, $\gamma_3 = 0.3$, $\gamma_4 = 0.4$, $\gamma_5 = 0.5$, $\gamma_6 = 0.6$, $t = 5.5$ and temperature distribution (on the right) by taking $\omega = 0.2$, $\alpha = 0.02$, $\Pr = 0.6$, $Br = 4$, $\gamma_1 = 0.1$, $\gamma_2 = 0.2$, $\gamma_3 = 0.3$, $\gamma_4 = 0.4$, $\gamma_5 = 0.5$, $\gamma_6 = 0.6$, $\beta = 0.5$, $t = 7$.

Figures 4 and 5. Influence of different time level on velocity profile (on the left) and temperature distribution (on the right).
Figures 6 and 7. Velocity distribution \( t \) of fluid at various level (on the left) when \( \omega = 0.2, \gamma_1 = 0.1, \gamma_2 = 0.2, \gamma_3 = 0.3, \gamma_4 = 0.4, \gamma_5 = 0.5, \gamma_6 = 0.6 \) and temperature distribution of fluid (on the right) by taking \( \omega = 0.2, \alpha = 0.02, Pr = 0.6, Br = 4, \gamma_1 = 0.1, \gamma_2 = 0.2, \gamma_3 = 0.3, \gamma_4 = 0.4, \gamma_5 = 0.5, \gamma_6 = 0.6, \beta = 0.5 \)

Figures 8 and 9. Effect of \( \gamma_1 \) on the velocity profile (on the left) when \( \gamma_5 = 0.2, \gamma_6 = 0.3, \gamma_1 = 0.2, \gamma_3 = 0.3, \gamma_4 = 0.2, \gamma_5 = 0.3, \gamma_6 = 0.4, \gamma_1 = 0.2, \gamma_2 = 0.3, \gamma_3 = 0.4, \gamma_4 = 0.5, \gamma_5 = 0.6, \beta = 0.5 \) and on temperature distribution (on the right) by taking \( \gamma_5 = 0.2, \gamma_6 = 0.3, \gamma_1 = 0.2, \gamma_2 = 0.3, \gamma_3 = 0.4, \gamma_4 = 0.5, \gamma_5 = 0.6, \gamma_6 = 0.7, \beta = 0.5, \alpha = 0.03 \)

Figures 10 and 11. Effect of \( \gamma_2 \) on the velocity (on the left) when \( \gamma_5 = 0.2, \gamma_6 = 0.3, \gamma_1 = 0.2, \gamma_3 = 0.3, \gamma_4 = 0.2, \gamma_5 = 0.3, \gamma_6 = 0.4, \gamma_1 = 0.2, \gamma_2 = 0.3, \gamma_3 = 0.4, \gamma_4 = 0.5, \gamma_5 = 0.6, \beta = 0.5, \alpha = 0.03 \)
Figures 12 and 13. Effect of $\gamma_3$ on the velocity (on the left) when $\gamma_2 = 0.4$, $\gamma_1 = 0.1$, $\gamma_2 = 0.6$, $\gamma_6 = 0.4$, $\gamma_7 = 0.3$, $\Phi = 0.4$ and temperature distribution (on the right) by taking $\gamma_2 = 0.2$, $\gamma_4 = 0.5$, $\gamma_5 = 0.4$, $\gamma_7 = 0.8$, $\gamma_6 = 0.5$, $\gamma_1 = 0.4$, $t = 5$, $\beta = 0.6$, $Br = 0.5$, $\alpha = 0.03$

Figures 14 and 15. Effect of $\gamma_4$ on the velocity (on the left) when $\gamma_2 = 0.1$, $\gamma_1 = 0.2$, $\gamma_2 = 0.6$, $\gamma_6 = 0.4$, $\gamma_7 = 0.3$, $\gamma_4 = 0.2$, $\Phi = 0.5$ and temperature distribution (on the right) by taking $\gamma_2 = 0.3$, $\gamma_4 = 0.5$, $\gamma_7 = 0.3$, $\gamma_6 = 0.7$, $\gamma_4 = 0.4$, $\gamma_1 = 0.8$, $t = 5$, $\beta = 0.6$, $Br = 0.5$, $\alpha = 0.04$

Figures 16 and 17. Effect of $\gamma_5$ on the velocity (on the left) when $\gamma_2 = 0.4$, $\gamma_1 = 0.1$, $\gamma_2 = 0.6$, $\gamma_6 = 0.4$, $\gamma_7 = 0.3$, $\gamma_5 = 0.3$, $\Phi = 0.4$ and temperature distribution (on the right) by taking $\gamma_2 = 0.2$, $\gamma_5 = 0.5$, $\gamma_7 = 0.8$, $\gamma_6 = 0.5$, $\gamma_1 = 0.4$, $\gamma_6 = 0.4$, $t = 5$, $\beta = 0.6$, $Br = 0.5$, $\alpha = 0.03$
Conclusion

Heat transferred analysis and unsteady thin film flow of an MHD third order fluid through a porous and oscillating belt has been discussed. The problems have been solved OHAM and HPM. Solutions at different time levels on velocity and temperature fields have been shown by taking the different values of the physical parameters. The results obtained from both methods are very identical. The effect of different nondimensional physical parameters are plotted and discussed.

References


Nomenclature

**Latin symbols**

- $A_1$-$A_3$ – kinematical tensors, [Nm$^{-2}$]
- $B_r$ – Brinkman number, [-]
- $c_p$ – specific heat, [kg$^{-1}$K$^{-1}$]
- $g$ – gravitational acceleration, [ms$^{-2}$]
- $K_1$ – Darcy permeability, [-]
- $M$ – magnetic parameter, [m$^{-1}$]
- $Pr$ – Prandtl number, [-]
- $pI$ – isotropic stress, [Nm$^{-2}$]
- $pS$ – spherical stress, [Nm$^{-2}$]
- $St$ – Stock number, [-]
- $T$ – Cauchy stress tensor, [Nm$^{-2}$]
- $v$ – velocity vector, [ms$^{-1}$]

**Greek symbols**

- $\alpha(i = 1, 2), \beta(j = 1, 2, 3)$ – material constants
- $\beta$ – non-Newtonian parameter
- $\gamma_i, \gamma_k$ – non-dimensional parameters
- $\delta$ – constant thickness, [m]
- $\theta$ – temperature distribution, [K]
- $\kappa$ – thermal conductivity
- $\mu$ – dynamic viscosity, [kgm$^{-1}$s$^{-1}$]
- $\rho$ – constant density, [kgm$^{-3}$]
- $\phi$ – non-dimensional porosity parameter
- $\phi$ – porosity, [-]
- $\omega$ – frequency of the oscillating belt, [s$^{-1}$]