LOCAL FRACTIONAL LAPLACE SERIES EXPANSION METHOD FOR DIFFUSION EQUATION ARISING IN FRACTAL HEAT TRANSFER

by

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In this paper, we first propose the local fractional Laplace series expansion method, which is a coupling method of series expansion method and Laplace transform via local fractional differential operator. An illustrative example for handling the diffusion equation arising in fractal heat transfer is given.

Key words: *analytical solution, diffusion equation, heat transfer, Laplace series expansion method, Laplace transform*

Introduction

Local fractional integral transforms have potential applications for science and engineering [1-4]. They was utilized to find the solutions for differential equations in the mathematical modeling of complex systems in engineering to capture the relations in space and time with the kernels within non-differentiability and irregular sets like fractals [5-11]. The local fractional Laplace transform (LFLT) was applied to couple other methods, such as decomposition method (DM) [5] and variational iteration method (VIM) [11-19]. Recently, the local fractional series expansion method (LFSEM) was suggested in [20] and developed to solve the differential equations within local fractional derivatives (LFD) [21, 22]. However, the coupling scheme of LFSEM with LFLT is not considered. The target of this paper is to present the local fractional Laplace series expansion method to deal with the diffusion equation arising in fractal heat transfer [23-25].

Fundamentals

The local fractional integral operator of $\omega(x)$ is defined as [1-15, 23]:

$$ {}_aI_b^{(\alpha)}\omega(\tau)=\frac{1}{\Gamma(1+\alpha)}\int_a^b \omega(\tau)(\mathrm{d}\tau)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta\tau\to 0}\sum_{j=0}^{j=N-1}\omega(\tau)(\Delta\tau)^\alpha \tag{1}$$

where $\Delta\tau=t_{j+1}-t_j$, $j=0,...,N-1$, $t_0=a$, $t_N=b$.

As the inverse operator of eq. (1), the local fractional derivative of $\Omega(\tau)$ is defined as [1-5, 23-25]:

$$\Omega^{(\alpha)}(\tau_0)=\frac{\mathrm{d}^\alpha\Omega(\tau)}{\mathrm{d}\tau^\alpha}\Big|_{\tau=\tau_0}=\lim_{\tau\to\tau_0}\frac{\Delta^\alpha[\Omega(\tau)-\Omega(\tau_0)]}{(\tau-\tau_0)^\alpha} \tag{2}$$

with $\Delta^\alpha[\Omega(\tau)-\Omega(\tau_0)]\cong\Gamma(1+\alpha)\Delta[\Omega(\tau)-\Omega(\tau_0)]$.

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The LFLT of $\Omega(\tau)$ is defined as [12-15]:

$$\widetilde{Y}_\alpha\{\Omega(\tau)\} = \Omega_y^{\widetilde{Y},\alpha}(y) = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha(-y^\alpha\tau^\alpha)\Omega(\tau)(\mathrm{d}\tau)^\alpha, \quad 0<\alpha\le 1 \tag{3}$$

The inverse LFLT of $\Omega(\tau)$ is defined as [12-15]:

$$\Omega(\tau) = \widetilde{Y}_\alpha^{-1}\{\Omega_y^{\widetilde{Y},\alpha}(y)\} = \frac{1}{(2\pi)^\alpha}\int_{\beta-i\infty}^{\beta+i\infty} E_\alpha(y^\alpha\tau^\alpha)\Omega_y^{\widetilde{Y},\alpha}(y)(\mathrm{d}y)^\alpha \tag{4}$$

where $y^\alpha = \beta^\alpha + i^\alpha\infty^\alpha$, and $\mathrm{Re}(y^\alpha) = \beta^\alpha$.

Some properties which are applied to this manuscript are [1, 4]:

$$\widetilde{Y}_\alpha\{a\Omega_1(\tau) + b\Omega_2(\tau)\} = a\widetilde{Y}_\alpha\{\Omega_1(\tau)\} + b\widetilde{Y}_\alpha\{\Omega_2(\tau)\} \tag{5}$$

$$\widetilde{Y}_\alpha\{\Omega^{(n\alpha)}(\tau)\} = y^{n\alpha}\widetilde{Y}_\alpha[\Omega(\tau)] - \sum_{k=1}^{n} y^{(k-1)\alpha}\Omega^{(n-k)\alpha}(0) \tag{6}$$

$$\widetilde{Y}_\alpha\{E_\alpha(x^\alpha)\} = \frac{1}{y^\alpha} \tag{7}$$

$$\widetilde{Y}_\alpha\left[\frac{\tau^{k\alpha}}{\Gamma(1+k\alpha)}\right] = \frac{1}{y^{(k+1)\alpha}} \tag{8}$$

Analysis of the method

We consider a given differential equation in local form:

$$\psi_\tau^{(\alpha)} = \mathrm{K}_\alpha\psi \tag{9}$$

where $\psi_\tau^{(\alpha)} = \partial^\alpha\psi(x,\tau)/\mathrm{d}\tau^\alpha$ and K_α is a linear local operator with respect to x.

We consider a multi-term separated functions of independent variables t and x, namely:

$$\psi(x,\tau) = \sum_{i=0}^{\infty}\sigma_i(\tau)\omega_i(x) \tag{10}$$

where $\sigma_i(\tau)$ and $\omega_i(x)$ are two local fractional continuous functions.

Setting $\sigma_i(\tau) = \tau^{i\alpha}/\Gamma(1+i\alpha)$, we have:

$$\psi(x,\tau) = \sum_{i=0}^{\infty}\frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)}\omega_i(x) \tag{11}$$

Taking the LFLT of eq. (11), we obtain:

$$\psi(y,\tau) = \sum_{i=0}^{\infty}\frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)}\omega_i(y) \tag{12}$$

Hence, we obtain:

$$\tilde{Y}_\alpha\{\psi_\tau^{(\alpha)}(x,\tau)\} = \sum_{i=0}^{\infty}\frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)}\tilde{Y}_\alpha\{\omega_{i+1}(x)\} = \sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\alpha)}t^{i\alpha}\omega_{i+1}(y) \tag{13}$$

$$\tilde{Y}_\alpha\{K_\alpha\psi(y,\tau)\} = K_\alpha\left[\sum_{i=0}^{\infty}\frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)}\omega_i(y)\right] = \sum_{i=0}^{\infty}\frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)}(K_\alpha\omega_i)(y) \tag{14}$$

Making use of eqs. (13) and (14), from eq. (9) one obtain:

$$\sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\alpha)}t^{i\alpha}\omega_{i+1}(y) = \sum_{i=0}^{\infty}\frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)}(K_\alpha\omega_i)(y) \tag{15}$$

which leads to the recursion:

$$\omega_{i+1}(y) = (K_\alpha\omega_i)(y) \tag{16}$$

Adopting the recursion formula (16), we have:

$$\psi(y,\tau) = \sum_{i=0}^{\infty}\frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)}\omega_i(y) \tag{17}$$

where the convergent condition reads:

$$\lim_{i\to\infty}\left[\frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)}\omega_i(y)\right] = 0 \tag{18}$$

Hence, the solution of eq. (9) is determined by:

$$\psi(\tau,y) = \tilde{Y}_\alpha^{-1}\{\psi(y,\tau)\} = \sum_{i=0}^{\infty}\frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)}\tilde{Y}_\alpha^{-1}\{\omega_i(y)\} \tag{19}$$

An analytical solution for diffusion equation arising in fractal heat transfer

We now consider the diffusion equation arising in fractal heat transfer [23-25]:

$$\psi_t^{(\alpha)}(x,\tau) - \psi_x^{(2\alpha)}(x,\tau) = 0, \quad 0<\alpha\le 1 \tag{20}$$

We present initial values as follows:

$$\psi(x,0) = E_\alpha(x^\alpha) \tag{21}$$

Adopting (16), we have:

$$\begin{cases}\omega_{i+1}(y) = (K_\alpha\omega_i)(y) = \dfrac{1}{y^\alpha}\\[2ex] \omega_0(y) = \tilde{Y}_\alpha\{\psi(x,0)\} = \tilde{Y}_\alpha\{E_\alpha(x^\alpha)\} = \dfrac{1}{y^\alpha}\end{cases} \tag{22}$$

such that the recurrence terms are written as:

$$\omega_1(y) = \frac{1}{y^\alpha} \tag{23}$$

$$\omega_2(y) = \frac{1}{y^\alpha} \tag{24}$$

$$\omega_3(y) = \frac{1}{y^\alpha} \tag{25}$$

and so on.

Hence, we get:

$$\psi(y,\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)} \frac{1}{y^\alpha} \tag{26}$$

Taking inverse LFLT, the non-differentiable solution of diffusion equation arising in fractal heat transfer can be written as:

$$\psi(x,\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)} E_\alpha(x^\alpha) = $$

$$= E_\alpha(x^\alpha) E_\alpha(\tau^\alpha) \tag{27}$$

and its graph is given in fig.1.

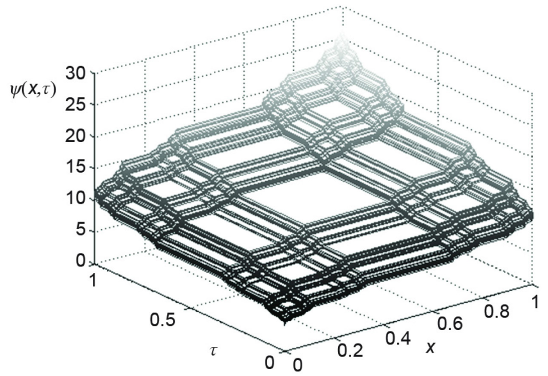

Figure 1. The non-differentiable solution of diffusion equation arising in fractal heat transfer when the fractal dimension α is equal to ln2/ln3

Conclusions

In this work, we first had proposed the coupling scheme of LFSEM with LFLT, which called local fractional Laplace series expansion method (LFLSEM). Based on it, we find the non-differentiable solution of diffusion equation arising in fractal heat transfer. The obtained result shows that the presented technology is easy, simple, efficient and accurate.

Nomenclature

x – space co-ordinates, [m]
$\tilde{Y}_\alpha[\Omega(\tau)]$ – LFLT of $\Omega(\tau)$, [–]
$\tilde{Y}_\alpha^{-1}[\Omega_y^{\tilde{Y},\alpha}(y)]$ – inverse LFLT of $\Omega_y^{\tilde{Y},\alpha}(y)$, [–]

Greek symbols

α – time fractal dimensional order, [–]
τ – time, [s]
$\psi(x, \tau)$ – concentration, [–]

References

[1] Yang, X. J., *Local Fractional Functional Analysis & Its Applications*, Asian Academic Publisher Limited, Hong Kong, 2011
[2] Yang, X.-J., *et al.*, *Local Fractional Integral Transforms and Applications*, Elsevier, 2015
[3] Srivastava, H. M., *et al.*, *Special Functions in Fractional Calculus and Related Fractional Differintegral Equations*, World Scientific, Singapore, 2015
[4] Yang, X. J. Local Fractional Integral Transforms, *Progress in Nonlinear Science, 4* (2011), 1, pp. 1-225
[5] Cattani, C., *et al.*, *Fractional Dynamics*, Emerging Science Publishers, 2015

[6] Zhong, W. P., *et al*., Applications of Yang-Fourier Transform to Local Fractional Equations with Local Fractional Derivative and Local Fractional Integral, *Advanced Materials Research, 461* (2012), March, pp. 306-310
[7] Yang, A. M., *et al*., The Yang-Fourier Transforms to Heat-Conduction in a Semi-Infinite Fractal Bar, *Thermal Science, 17* (2013), 3, pp. 707-713
[8] Yang, X. J., *et al*., A Novel Approach to Processing Fractal Signals Using the Yang-Fourier Transforms, *Procedia Engineering, 29* (2012), Feb., pp. 2950-2954
[9] Yang, X. J., *et al*., Mathematical Aspects of the Heisenberg Uncertainty Principle within Local Fractional Fourier Analysis, *Boundary Value Problems, 2013* (2013), 1, pp. 1-16
[10] Wang, S. Q., *et al*., Local Fractional Function Decomposition Method for Solving Inhomogeneous Wave Equations with Local Fractional Derivative, *Abstract and Applied Analysis, 2014* (2014), ID 176395
[11] Yang, X. J., Local Fractional Partial Differential Equations with Fractal Boundary Problems, *Advances in Computational Mathematics and its Applications, 1* (2012), 1, pp. 60-63
[12] He, J.-H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, *International Journal of Theoretical Physics, 53* (2014), 11, pp. 3698-3718
[13] Zhang, Y. Z., *et al*., Initial Boundary Value Problem for Fractal Heat Equation in the Semi-Infinite Region by Yang-Laplace Transform, *Thermal Science, 18* (2014), 2, pp. 677-681
[14] Zhao, Y., *et al*., Mappings for Special Functions on Cantor Sets and Special Integral Transforms via Local Fractional Operators, *Abstract and Applied Analysis, 2013* (2013), ID 316978
[15] Zhao, C. G., *et al*., The Yang-Laplace Transform for Solving the IVPs with Local Fractional Derivative, *Abstract and Applied Analysis, 2014* (2014), ID 386459
[16] Liu, C. F., *et al*., Reconstructive Schemes for Variational Iteration Method within Yang-Laplace Transform with Application to Fractal Heat Conduction Problem, *Thermal Science, 17* (2013), 3, pp. 715-721
[17] Yang, A. M., *et al*., Local Fractional Laplace Variational Iteration Method for Solving Linear Partial Differential Equations with Local Fractional Derivative, *Discrete Dynamics in Nature and Society, 2014* (2014), ID 365981
[18] Li, Y., *et al*., Local Fractional Laplace Variational Iteration Method for Fractal Vehicular Traffic Flow, *Advances in Mathematical Physics, 2014* (2014), ID 649318
[19] Xu, S., *et al*., Local Fractional Laplace Variational Iteration Method for Nonhomogeneous Heat Equations Arising in Fractal Heat Flow, *Mathematical Problems in Engineering, 2014* (2014), ID 914725
[20] Yang, A. M., *et al*., Local Fractional Series Expansion Method for Solving Wave and Diffusion Equations on Cantor Sets, *Abstract and Applied Analysis, 2013* (213), ID 351057
[21] Zhao, Y., *et al*., Approximation Solutions for Local Fractional Schrödinger Equation in the One-Dimensional Cantorian System, *Advances in Mathematical Physics, 2013* (2013), ID 291386
[22] Yang, A. M., *et al*., Application of Local Fractional Series Expansion Method to Solve Klein-Gordon Equations on Cantor Sets, *Abstract and Applied Analysis, 2014* (2014), ID 372741
[23] Yang, X. J., *Advanced Local Fractional Calculus and its Applications*, World Science, New York, USA, 2012
[24] Yang, X. J., *et al*., Approximate Solutions for Diffusion Equations on Cantor Space-Time, *Proceedings of the Romanian Academy, Series A, 14* (2013), 2, pp. 127-133
[25] Hao, Y. J., *et al.,* Helmholtz and Diffusion Equations Associated with Local Fractional Derivative Operators Involving the Cantorian and Cantor-Type Cylindrical Coordinates, *Advances in Mathematical Physics, 2013* (2013), ID 754248

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