LOCAL FRACTIONAL LAPLACE SERIES EXPANSION METHOD
FOR DIFFUSION EQUATION ARISING IN FRACTAL HEAT TRANSFER

by

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In this paper, we first propose the local fractional Laplace series expansion method, which is a coupling method of series expansion method and Laplace transform via local fractional differential operator. An illustrative example for handling the diffusion equation arising in fractal heat transfer is given.

Key words: analytical solution, diffusion equation, heat transfer, Laplace series expansion method, Laplace transform

Introduction

Local fractional integral transforms have potential applications for science and engineering [1-4]. They were utilized to find the solutions for differential equations in the mathematical modeling of complex systems in engineering to capture the relations in space and time with the kernels within non-differentiability and irregular sets like fractals [5-11]. The local fractional Laplace transform (LFLT) was applied to couple other methods, such as decomposition method (DM) [5] and variational iteration method (VIM) [11-19]. Recently, the local fractional series expansion method (LFSEM) was suggested in [20] and developed to solve the differential equations within local fractional derivatives (LFD) [21, 22]. However, the coupling scheme of LFSEM with LFLT is not considered. The target of this paper is to present the local fractional Laplace series expansion method to deal with the diffusion equation arising in fractal heat transfer [23-25].

Fundamentals

The local fractional integral operator of $\omega(x)$ is defined as [1-15, 23]:

$$a I_b^{(\alpha)}\omega(\tau) = \frac{1}{\Gamma(1+\alpha)} \int_a^b \omega(\tau)(d\tau)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta\tau \to 0} \sum_{j=0}^{N-1} \omega(\tau)(\Delta\tau)^\alpha$$

where $\Delta\tau = t_{j+1} - t_j$, $j = 0, ..., N - 1$, $t_0 = a$, $t_N = b$.

As the inverse operator of eq. (1), the local fractional derivative of $\Omega(\tau)$ is defined as [1-5, 23-25]:

$$\Omega^{(\alpha)}(\tau_0) = \frac{d^{\alpha}\Omega(\tau)}{d\tau^{\alpha}} \bigg|_{\tau = \tau_0} = \lim_{\tau \to \tau_0} \frac{\Delta^{\alpha}[\Omega(\tau) - \Omega(\tau_0)]}{(\tau - \tau_0)^{\alpha}}$$

with $\Delta^{\alpha}[\Omega(\tau) - \Omega(\tau_0)] \equiv \Gamma(1+\alpha)\Delta[\Omega(\tau) - \Omega(\tau_0)]$.

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The LFLT of $\Omega(\tau)$ is defined as [12-15]:

$$\bar{Y}_a \{\Omega(\tau)\} = \Omega^\alpha \varphi \left( y \right) = \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty E_a \left( -y^\alpha \tau^\alpha \right) \Omega(\tau)(d\tau)^\alpha, \quad 0 < \alpha \leq 1 \quad (3)$$

The inverse LFLT of $\Omega(\tau)$ is defined as [12-15]:

$$\Omega(\tau) = Y_a^{-1} \{\Omega^\alpha \varphi \left( y \right)\} = \frac{1}{(2\pi)^\alpha} \int_{\beta-i\infty}^{\beta+i\infty} E_a \left( -y^\alpha \tau^\alpha \right) \Omega^\alpha \varphi \left( y \right)(dy)^\alpha \quad (4)$$

where $y^\alpha = \beta^\alpha + i^\alpha \alpha^\alpha$, and $\text{Re}(y^\alpha) = \beta^\alpha$.

Some properties which are applied to this manuscript are [1, 4]:

$$Y_a \{a\Omega_1(\tau) + b\Omega_2(\tau)\} = aY_a \{\Omega_1(\tau)\} + bY_a \{\Omega_2(\tau)\} \quad (5)$$

$$Y_a \{\Omega^{\alpha \alpha}(\tau)\} = \eta^{\alpha \alpha} \varphi(Y_a[\Omega(\tau)] - \sum_{k=1}^n \eta^{(k-1)\alpha} \Omega^{\alpha \alpha}(0) \quad (6)$$

$$Y_a \{E_a (x^\alpha)\} = \frac{1}{y^\alpha} \quad (7)$$

$$Y_a \left[ \frac{\tau^\alpha}{\Gamma(1+k\alpha)} \right] = \frac{1}{y^{(k+1)\alpha}} \quad (8)$$

**Analysis of the method**

We consider a given differential equation in local form:

$$\psi^{\alpha \alpha} = K_a \psi \quad (9)$$

where $\psi^{\alpha \alpha} = \delta^{\alpha \alpha} \psi(x, \tau)/d\tau^\alpha$ and $K_a$ is a linear local operator with respect to $x$.

We consider a multi-term separated functions of independent variables $t$ and $x$,

$$\psi(x, \tau) = \sum_{i=0}^\infty \sigma_i(\tau) \omega_i(x) \quad (10)$$

where $\sigma_i(\tau)$ and $\omega_i(x)$ are two local fractional continuous functions.

Setting $\sigma_i(\tau) = \tau^{\alpha \alpha} / \Gamma(1+i\alpha)$, we have:

$$\psi(x, \tau) = \sum_{i=0}^\infty \frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)} \omega_i(x) \quad (11)$$

Taking the LFLT of eq. (11), we obtain:

$$\psi(y, \tau) = \sum_{i=0}^\infty \frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)} \omega_i(y) \quad (12)$$

Hence, we obtain:
\[ Y_\alpha \{ \psi_1^\alpha (x, \tau) \} = \sum_{i=0}^{\infty} \frac{\psi_1^\alpha(i)}{\Gamma(1 + i\alpha)} Y_\alpha \{ \psi_{i+1}(x) \} = \sum_{i=0}^{\infty} \frac{1}{\Gamma(1 + i\alpha)} \psi_1^\alpha(y) \]  
(13)

\[ \tilde{Y}_\alpha \{ K_\alpha \psi(y, \tau) \} = K_\alpha \left[ \sum_{i=0}^{\infty} \frac{\psi_1^\alpha(i)}{\Gamma(1 + i\alpha)} \omega_i(y) \right] = \sum_{i=0}^{\infty} \frac{\psi_1^\alpha(i)}{\Gamma(1 + i\alpha)} (K_\alpha \omega_i)(y) \]  
(14)

Making use of eqs. (13) and (14), from eq. (9) one obtain:

\[ \sum_{i=0}^{\infty} \frac{1}{\Gamma(1 + i\alpha)} \psi_1^\alpha(y) \omega_{i+1}(y) = \sum_{i=0}^{\infty} \frac{\psi_1^\alpha(i)}{\Gamma(1 + i\alpha)} (K_\alpha \omega_i)(y) \]  
(15)

which leads to the recursion:

\[ \omega_{i+1}(y) = (K_\alpha \omega_i)(y) \]  
(16)

Adopting the recursion formula (16), we have:

\[ \psi(y, \tau) = \sum_{i=0}^{\infty} \frac{\psi_1^\alpha(i)}{\Gamma(1 + i\alpha)} \omega_i(y) \]  
(17)

where the convergent condition reads:

\[ \lim_{\alpha \to \infty} \left[ \frac{\psi_1^\alpha(i)}{\Gamma(1 + i\alpha)} \omega_i(y) \right] = 0 \]  
(18)

Hence, the solution of eq. (9) is determined by:

\[ \psi(\tau, y) \psi^{-1} \{ \psi(y, \tau) \} = \sum_{i=0}^{\infty} \frac{\psi_1^\alpha(i)}{\Gamma(1 + i\alpha)} \psi^{-1} \{ \psi(y, 0) \} \]  
(19)

An analytical solution for diffusion equation arising in fractal heat transfer

We now consider the diffusion equation arising in fractal heat transfer [23-25]:

\[ \psi_1^\alpha(x, \tau) - \psi_2^\alpha(x, \tau) = 0, \quad 0 < \alpha \leq 1 \]  
(20)

We present initial values as follows:

\[ \psi(x, 0) = E_\alpha(x^\alpha) \]  
(21)

Adopting (16), we have:

\[ \begin{cases} 
\omega_{i+1}(y) = (K_\alpha \omega_i)(y) = \frac{1}{y^\alpha} \\
\omega_0(y) = \tilde{Y}_\alpha \{ \psi(x, 0) \} = \tilde{Y}_\alpha \{ E_\alpha(x^\alpha) \} = \frac{1}{y^\alpha} 
\end{cases} \]  
(22)

such that the recurrence terms are written as:
\[ \omega_1(y) = \frac{1}{y^\alpha} \]  \hspace{2cm} (23)

\[ \omega_2(y) = \frac{1}{y^\alpha} \]  \hspace{2cm} (24)

\[ \omega_3(y) = \frac{1}{y^\alpha} \]  \hspace{2cm} (25)

and so on.

Hence, we get:

\[ \psi(y, \tau) = \sum_{i=0}^{\infty} \frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)} \frac{1}{y^\alpha} \]  \hspace{2cm} (26)

Taking inverse LFLT, the non-differentiable solution of diffusion equation arising in fractal heat transfer can be written as:

\[ \psi(x, \tau) = \sum_{i=0}^{\infty} \frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)} E_{\alpha}(x^{\alpha}) = \]

\[ = E_{\alpha}(x^{\alpha})E_{\alpha}(\tau^{\alpha}) \]  \hspace{2cm} (27)

and its graph is given in fig.1.

**Conclusions**

In this work, we first had proposed the coupling scheme of LFSEM with LFLT, which called local fractional Laplace series expansion method (LFLSEM). Based on it, we find the non-differentiable solution of diffusion equation arising in fractal heat transfer. The obtained result shows that the presented technology is easy, simple, efficient and accurate.

**Nomenclature**

- \( x \) – space co-ordinates, [m]
- \( \mathcal{Y}_\alpha[\Omega(t)] \) – LFLT of \( \Omega(t) \), [-]
- \( \mathcal{Y}_\alpha^{-1}[\Omega(y)^\alpha] \) – inverse LFLT of \( \Omega(y)^\alpha \), [-]

**Greek symbols**

- \( \alpha \) – time fractal dimensional order, [-]
- \( \tau \) – time, [s]
- \( \psi(x, \tau) \) – concentration, [-]

**References**


