

GENERALIZED EXP-FUNCTION METHOD FOR NON-LINEAR SPACE-TIME FRACTIONAL DIFFERENTIAL EQUATIONS

by

Li-Mei YAN* and Feng-Sheng XU

School of Mathematical Sciences, Dezhou University, Dezhou, China

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A generalized exp-function method is proposed to solve non-linear space-time fractional differential equations. The basic idea of the method is to convert a fractional partial differential equation into an ordinary equation with integer order derivatives by fractional complex transform. To illustrate the effectiveness of the method, space-time fractional asymmetrical Nizhnik-Novikov-Veselov equation is considered. The fractional derivatives in the present paper are in Jumarie's modified Riemann-Liouville sense.

Key words: *generalized exp-function method, non-linear space-time fractional differential equation, modified Riemann-Liouville derivative, asymmetrical Nizhnik-Novikov-Veselov equation*

Introduction

Approximate solutions or exact solutions play an important role in the study of fractional differential equations being their extensive applications in engineering and many other applied fields. By now, many effective methods for solving fractional differential equations have been proposed by many researchers such as homotopy perturbation method [1, 2], variational iteration method [3], fractional sub-equation method [4, 5], iterative Laplace transform method [6, 7], and so on.

It is well known that exp-function method proposed by He [8] in 2006 is a very powerful tool for solving non-linear partial differential equations. The method has been successfully applied to search for solutions of many non-linear equations. The motivation of this paper is to extend the exp-function method to solve non-linear space-time fractional differential equations with Jumarie's modified Riemann-Liouville derivatives.

Modified Riemann-Liouville derivative and Generalized exp-function method

The Jumarie's modified Riemann-Liouville derivative of fractional order α of function $f(x)$ is defined as [9]:

$$D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} [f(\xi) - f(0)] d\xi, & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} [f(\xi) - f(0)] d\xi, & 0 < \alpha < 1 \\ [D_x^{(\alpha-n)} f(x)]^{(n)}, & n \leq \alpha < n+1, n \geq 1 \end{cases} \quad (1)$$

* Corresponding author; e-mail: yanlimei2013@163.com

Several useful properties of Jumarie's modified Riemann-Liouville derivative are listed in [9]:

$$D_x^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0 \quad (2)$$

$$[u(x)v(x)]^{(\alpha)} = u(x)^{(\alpha)}v(x) + u(x)v(x)^{(\alpha)} \quad (3)$$

$$\{f[u(x)]\}^{(\alpha)} = f'_u(u)u^{(\alpha)}(x) = f_u^{(\alpha)}(u)(u'_x)^\alpha \quad (4)$$

A generalized exp-function method is proposed to solve non-linear space-time fractional differential equations. The main steps of the method are.

Step 1. Consider the non-linear space-time fractional differential equation:

$$P(u, D_t^\alpha u, D_x^\beta u, D_t^{2\alpha} u, D_x^{2\beta} u, \dots) = 0, \quad 0 < \alpha, \beta \leq 1 \quad (5)$$

where x and t are independent variables, and u is dependent variable, $D_t^\alpha u, D_x^\beta u, D_t^{2\alpha} u$, and $D_x^{2\beta} u$ are Jumarie's modified Riemann-Liouville derivatives of u with respect to t and x , P is a polynomial about u and its derivatives in which the highest order derivative term and non-linear term are involved.

Step 2. Making fractional transform:

$$\eta = k \frac{x^\beta}{\Gamma(1+\beta)} + \omega \frac{t^\alpha}{\Gamma(1+\alpha)} \quad (6)$$

Equation (5) can be reduced to its ordinary counterpart:

$$P(u, \omega u', k u'', \omega^2 u''', k^2 u''', \dots) = 0 \quad (7)$$

Step 3. Assume that the solution of eq. (7) can be expressed:

$$u = \frac{\sum_{i=-c}^d a_i \exp(i\eta)}{\sum_{j=-p}^q b_j \exp(j\eta)} \quad (8)$$

where c, d, p , and q are positive integers to be determined by homogeneous balance method, and a_i and b_i are constants to be determined later.

Step 4. By balancing the highest order derivative term and non-linear term in eq. (7), one gets c, d, p , and q .

Step 5. Substituting eq. (8) into eq. (7) and equating the coefficients of $\exp(i\eta)$ to zero, one gets a system of algebraic equations about the unknown constants a_i and b_i . Solving the obtained algebraic equations with the help of Mathematica, one obtains the exact solutions of eq. (5) eventually.

The fractional asymmetrical Nizhnik-Novikov-Veselov equation

To explain the effectiveness of the method, let us consider space-time fractional asymmetrical Nizhnik-Novikov-Veselov equation [10]:

$$\begin{cases} D_t^\alpha u + D_x^{3\beta} u - 3u D_x^\beta v - 3v D_x^\beta u = 0 \\ D_x^\beta u = D_y^\beta v, \quad 0 < \alpha, \beta \leq 1 \end{cases} \quad (9)$$

Making fractional complex transform:

$$u = u(\eta), \quad v = v(\eta), \quad \eta = l \frac{x^\beta}{\Gamma(1+\beta)} + m \frac{y^\beta}{\Gamma(1+\beta)} + n \frac{t^\alpha}{\Gamma(1+\alpha)} \quad (10)$$

Equation (9) can be converted into the form:

$$\begin{cases} l^3 u''' - \frac{3l^2}{m} u^2 + nu = 0 \\ lu = mv \end{cases} \quad (11)$$

In accordance with the generalized exp-function method, put:

$$u(\eta) = \frac{a_c \exp(c\eta) + \dots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \dots + b_{-q} \exp(-q\eta)} \quad (12)$$

By substituting eq. (12) into eq. (11) and balancing the highest order term $l^3 u'''$ and the non-linear term $(3l^2/m)u^2$ of eq. (11) results $p = c, q = d$.

For simplicity, we only discuss the case that $p = c = 1, q = d = 1$ here.

Put:

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)} \quad (13)$$

Substituting eq. (13) into eq. (11), we have:

$$\begin{aligned} & \frac{1}{A_1} [c_3 \exp(3\eta) + c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + \\ & + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) + c_{-3} \exp(-3\eta)] = 0 \end{aligned} \quad (14)$$

where $A_1 = [\exp(\eta) + b_0 + b_{-1} \exp(-\eta)]^3$.

Equating the coefficients of $\exp(i\eta)$, $i = -3, -2, -1, 0, 1, 2, 3$, to zero, we obtain a system of equations:

$$c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 0 \quad (15)$$

Solving the equations with the help of Mathematica, we have the results:

$$n = \frac{3a_{-1}l^2}{mb_{-1}}, \quad a_0 = \frac{a_{-1}b_0}{b_{-1}}, \quad a_1 = \frac{a_{-1}}{b_{-1}} \quad (16)$$

where $l, m, a_{-1}, b_{-1}, b_0$ are free parameters.

Therefore, eq. (9) has exact solution of the form:

$$u(\eta) = \frac{\frac{a_{-1}}{b_{-1}} \exp(\eta) + \frac{a_{-1}b_0}{b_{-1}} + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)} \quad (17)$$

Put $l = iL, m = iM, n = iN$ in which i is the imaginary unit, we have periodic solution of eq. (9):

$$u(\eta) = \frac{2a_{-1} \cos \left[L \frac{x^\beta}{\Gamma(1+\beta)} + M \frac{y^\beta}{\Gamma(1+\beta)} + N \frac{t^\alpha}{\Gamma(1+\alpha)} \right] + a_{-1}b_0}{2 \cos \left[L \frac{x^\beta}{\Gamma(1+\beta)} + M \frac{y^\beta}{\Gamma(1+\beta)} + N \frac{t^\alpha}{\Gamma(1+\alpha)} \right] + b_0} \quad (18)$$

where $b_{-1} = 1$, and $N = -(3a_{-1}L^2)/M$.

As the fractional orders α and β approach 1, the solutions (17) and (18) of eq. (9) is reduced to traveling wave solution of the classical asymmetrical Nizhnik-Novikov-Veselov equation obtained by exp-function method. So, the method is a generalization form of exp-function method.

Conclusions

A generalized exp-function method is firstly applied to solve non-linear fractional evolution equations. The obtained exact solutions by the method can be reduced to traveling wave solutions when the fractional orders α and β approach to 1. It is evident that the method is simple and straightforward to find exact solutions of many non-linear fractional differential equations.

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