

LIE SYMMETRY GROUP OF (2+1)-DIMENSIONAL JAULENT-MIODEK EQUATION

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In this paper, we consider a system of (2+1)-dimensional non-linear model by using auxiliary equation method and Clarkson-Kruskal direct method which is very important in fluid and physics. We construct some new exact solutions of (2+1)-dimensional non-linear models with the aid of symbolic computation which can illustrate some actions in fluid in the future.

Key words: *symmetry, exact traveling wave solutions, Jaulent-Miodek equation, Clarkson-Kruskal direct method*

Introduction

Non-linear evolution equation plays an important role in applied mathematics and physics. In recent years, various effective methods have been developed to find the exact solutions of non-linear partial differential equations. These methods include tanh function method [1], generalized hyperbolic function method [2], homogeneous balance method [3], Jacobi elliptic function expansion method [4], exponential function method [5], auxiliary equation method [6], Clarkson–Kruskal (CK) direct method [7, 8] and so on. The purpose of this work is to generalize the work made in [9, 10]. We apply this method to the (2+1)-dimensional Jaulent-Miodek equation which associates with energy-dependent Schrödinger potential and has many interesting characters. It is an important model in fluid and physics.

New exact traveling wave solutions for (2+1)-dimensional Jaulent-Miodek equation

In [11], Geng *et al.* developed some non-linear models generated by the Jaulent-Miodek hierarchy [12]. Wu [13] gave the N-soliton solution of the first model by using the Hirota bilinear method. Liu *et al.* [14] discussed the bifurcation and exact travelling wave solutions of the third one. Wazwaz [15] obtained the Multiple kink solutions and multiple singular kink solutions of the third model. We have studied the second model in [16] and obtained the multiple kink solutions. In this paper, we shall discuss the following (2+1)-dimensional Jaulent-Miodek equation:

$$w_t = -\frac{1}{4}(w_{xx} - 2w^3)_x - \frac{3}{4}\left(\frac{1}{4}\partial_x^{-1}w_{yy} + w_x\partial_x^{-1}w_y\right) \quad (1)$$

By substituting $w = u_x$ into (1), we can omit the integral term in eq. (1):

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$$u_{xt} + \frac{1}{4}u_{xxxx} - \frac{3}{2}u_x^2 u_{xx} + \frac{3}{16}u_{yy} + \frac{3}{4}u_{xx}u_y = 0 \quad (2)$$

To seek the travelling wave solution of eq. (2), we introduce:

$$u = U(\xi), \quad \xi = x + dy + et \quad (3)$$

where d and e are arbitrary constants. Substituting (3) into eq. (2) and integrating once with respect to ξ and setting the integration constant equal to zero, one has:

$$4U_{\xi\xi\xi} + 16dU_{\xi} - 8U_{\xi}^3 + 3e^2U_{\xi} + 6U_{\xi}^2e = 0$$

By introducing another transformation:

$$V(\xi) = U_{\xi} \quad (4)$$

we can arrive:

$$4V_{\xi\xi} + 16dV - 8V^3 + 3e^2V + 6V^2e = 0 \quad (5)$$

Balancing the linear terms of the highest order with the non-linear terms yields the leading order $m = 1$. If we propose that V has the form:

$$V(\xi) = a_0 + a_1z(\xi) \quad (6)$$

where a_0 , and a_1 are constants to be determined, and $z(\xi)$ express the solutions of [9, 10]:

$$\frac{dz(\xi)}{d\xi} = \sqrt{az^2(\xi) + bz^3(\xi) + cz^4(\xi)} \quad (7)$$

Substituting eqs. (7) and (6) into eq. (5) and setting the coefficients of $z^i(\xi)$ ($i = 0, 1, 2, 3$) to zero, we obtain following set of non-linear algebraic equations:

$$-8a_0^3 + 16da_0 + 3e^2a_0 + 6ea_0^2 = 0$$

$$-8a_1^3 + 8a_1c = 0$$

$$4a_1a - 24a_0^2a_1 + 16da_1 + 3e^2a_1 + 12ea_0a_1 = 0$$

$$6a_1b - 24a_0a_1^2 + 6ea_1^2 = 0$$

Solving this set of algebraic equations with the aid of Maple, we obtain:

$$a = -4d - \frac{3}{4}e^2, \quad a_0 = 0, \quad b = -ea_1, \quad c = a_1^2 \quad (8)$$

$$a = 8d + \frac{3}{2}e^2 + \frac{3}{4}e \left[\frac{3}{4}e \pm \frac{1}{4}\sqrt{33e^2 + 128d} \right], \quad a_0 = \frac{3}{8}e \pm \frac{1}{8}\sqrt{33e^2 + 128d},$$

$$b = a_1 \left(\frac{3}{2}e \pm \frac{1}{2}\sqrt{33e^2 + 128d} \right), \quad c = a_1^2 \quad (9)$$

Substituting eq. (8) with $z(\xi)$ in [10] into eq. (6) gives the exact solution of eq. (5):

$$V_1(\xi) = \frac{a_1 \left(4d + \frac{3}{4}e^2\right) \operatorname{sech}^2 \left(\frac{\sqrt{-4d - \frac{3}{4}e^2}}{2} \xi \right)}{-ea_1 + 2\epsilon \sqrt{-a_1^2 \left(4d + \frac{3}{4}e^2\right)} \tanh \left(\frac{\sqrt{-4d - \frac{3}{4}e^2}}{2} \xi \right)}$$

$$V_2(\xi) = \frac{-a_1 \left(4d + \frac{3}{4}e^2\right) \operatorname{csch}^2 \left(\frac{\sqrt{-4d - \frac{3}{4}e^2}}{2} \xi \right)}{-ea_1 + 2\epsilon \sqrt{-a_1^2 \left(4d + \frac{3}{4}e^2\right)} \coth \left(\frac{\sqrt{-4d - \frac{3}{4}e^2}}{2} \xi \right)}$$

where $\xi = x + dy + et$, and a_1, d , and e are arbitrary constants and $4d + (3/4)e^2 < 0$,

$$V_3(\xi) = \frac{a_1 \left(4d + \frac{3}{4}e^2\right) \sec^2 \left(\frac{\sqrt{4d + \frac{3}{4}e^2}}{2} \xi \right)}{-ea_1 + 2\epsilon \sqrt{a_1^2 \left(4d + \frac{3}{4}e^2\right)} \tan \left(\frac{\sqrt{4d + \frac{3}{4}e^2}}{2} \xi \right)}$$

$$V_4(\xi) = \frac{a_1 \left(4d + \frac{3}{4}e^2\right) \csc^2 \left(\frac{\sqrt{4d + \frac{3}{4}e^2}}{2} \xi \right)}{-ea_1 + 2\epsilon \sqrt{a_1^2 \left(4d + \frac{3}{4}e^2\right)} \cot \left(\frac{\sqrt{4d + \frac{3}{4}e^2}}{2} \xi \right)}$$

where $\xi = x + dy + et$, and a_1, d , and e are arbitrary constants and $4d + (3/4)e^2 > 0$.

By using eq. (4), we obtain:

$$U_1 = \frac{1}{\epsilon} \left(\ln -ea_1 \tanh^2 \left[\frac{\sqrt{-\left(4d + \frac{3}{4}e^2\right)}}{4} \xi \right] - ea_1 + 4\epsilon \sqrt{-a_1^2 \left(4d + \frac{3}{4}e^2\right)} \tanh \left[\frac{\sqrt{-\left(4d + \frac{3}{4}e^2\right)}}{4} \xi \right] \right) +$$

$$+ \frac{1}{\epsilon} \left(\ln \left\{ \tanh^2 \left[\frac{\sqrt{-\left(4d + \frac{3}{4}e^2\right)}}{4} \xi \right] + 1 \right\} \right)$$

$$U_2 = -\frac{1}{\epsilon} \left(\ln \left\{ -ea_1 \tanh \left[\frac{\sqrt{-\left(4d + \frac{3}{4}e^2\right)}}{4} \xi \right] + \right. \right.$$

$$\begin{aligned}
& + \epsilon \sqrt{-a_1^2 \left(4d + \frac{3}{4}e^2\right)} \left\{ \tanh^2 \left[\frac{\sqrt{-\left(4d + \frac{3}{4}e^2\right)}}{4} \xi \right] + 1 \right\} + \frac{1}{\epsilon} \left(\ln \left\{ \tanh \left[\frac{\sqrt{-\left(4d + \frac{3}{4}e^2\right)}}{4} \xi \right] \right\} \right) \\
U_3 &= \frac{1}{\epsilon} \ln \left\{ -ea_1 + 2\epsilon \sqrt{a_1^2 \left(4d + \frac{3}{4}e^2\right)} \tan \left[\frac{\sqrt{\left(4d + \frac{3}{4}e^2\right)}}{2} \xi \right] \right\} \\
U_4 &= \frac{1}{\epsilon} \left(\ln \left\{ \operatorname{tg} \left[\frac{\sqrt{\left(4d + \frac{3}{4}e^2\right)}}{2} \xi \right] \right\} - \ln \left\{ -ea_1 \tan \left[\frac{\sqrt{\left(4d + \frac{3}{4}e^2\right)}}{2} \xi \right] + 2\epsilon \sqrt{a_1^2 \left(4d + \frac{3}{4}e^2\right)} \right\} \right)
\end{aligned}$$

and solution of eq. (1) with $w = u_x$. Substituting eq. (9) with $z(\xi)$ in [11] into eq. (6) yields the exact traveling wave solution of eq. (5) as the same way as the former case.

Symmetry group transformation of the (2+1)-dimensional Jaulent-Miodek equation and its new exact solutions

The application of Lie theory has been playing an important role since it was demonstrated by Lie [17]. In 1989, Clarkson and Kruskal [18] developed the CK direct method that can be used to find symmetry reduction. Recently, Lou and Ma [19] introduced a modified CK direct method to obtain symmetry transformation group of a given PDE.

To get symmetry transformation of eq. (2), we suppose:

$$u = \alpha + \beta U(\xi, \eta, \tau) \quad (10)$$

where $\alpha = \alpha(x, y, t)$, $\beta = \beta(x, y, t)$, $\xi = \xi(x, y, t)$, $\eta = \eta(x, y, t)$, $\tau = \tau(x, y, t)$ are arbitrary functions of x, y , and t to be determined by restricting $U(\xi, \eta, \tau)$ to satisfy the same equation as u under the transformation $\{u, x, y, t\} \rightarrow \{U, \xi, \eta, \tau\}$.

Restrict U to satisfy the same equation as u , say:

$$U_{\xi\tau} + \frac{1}{4}U_{\xi\xi\xi\xi} - \frac{3}{2}U_{\xi}^2U_{\xi\xi} + \frac{3}{16}U_{\eta\eta} + \frac{3}{4}U_{\xi\xi}U_{\eta} = 0 \quad (11)$$

Substituting eq. (10) with eq. (11) into eq. (2) and setting the coefficients of U and its derivatives equal to zero, we arrive at some equations to be determined. After some tedious calculation, we have:

$$\xi = \tau_t^{1/3}x - \frac{8}{9}y^2 \frac{\tau_{tt}}{\tau_t^{2/3}} - \frac{8}{3} \frac{p_t}{\tau_t^{1/3}}y + q, \quad \eta = \tau_t^{2/3}y + p, \quad \beta = 1, \quad \tau = \tau(t) \quad (12)$$

$$\begin{aligned}
\alpha &= -\frac{2}{3} \frac{p_t x}{\tau_t^{2/3}} - \frac{4}{9} \frac{\tau_{tt} xy}{\tau_t} - \frac{32}{27} \frac{p_t \tau_{tt} y^2}{\tau_t^{5/3}} - \frac{8}{9} \frac{p_t^2 y}{\tau_t^{4/3}} - \frac{32}{81} \frac{\tau_{tt}^2 y^3}{\tau_t^2} + \\
&+ \frac{32}{81} \frac{\tau_{ttt} y^3}{\tau_t} - \frac{4}{3} \frac{q_t y}{\tau_t^{1/3}} + \frac{16}{9} \frac{p_{tt} y^2}{\tau_t^{2/3}} + m(t)
\end{aligned} \quad (13)$$

where $p = p(t)$, $q = q(t)$, $\tau = \tau(t)$ and $m(t)$ are the arbitrary function of t .

By using eq. (12), we have:

$$u = \alpha + \beta U(\xi, \eta, \tau)$$

where $\alpha, \beta, \xi, \eta, \tau$ are determined by eq. (12) and (13).

We also get the symmetry:

$$\begin{aligned} \sigma(U) = & \left(-\frac{8}{9}y^2 f_{tt} - \frac{8}{3}n_t y + h + \frac{1}{3}f_t x \right) U_x + \left(n + \frac{2}{3}y f_t \right) U_y + f U_t + s - \frac{4}{9}f_{tt} xy + \\ & + \frac{32}{81}y^3 f_{ttt} - \frac{2}{3}x n_t - \frac{4}{3}y h_t + \frac{16}{9}y^2 n_{tt} \end{aligned}$$

The general elements of Lie algebra can be written as:

$$V = V_1(f) + V_2(h) + V_3(n) + V_4(s)$$

$$V_1(f) = \left(-\frac{8}{9}y^2 f_{tt} + \frac{1}{3}f_t x \right) \frac{\partial}{\partial x} + \frac{2}{3}y f_t \frac{\partial}{\partial y} + f \frac{\partial}{\partial t} + \left(-\frac{4}{9}f_{tt} xy + \frac{32}{81}y^3 f_{ttt} \right) \frac{\partial}{\partial U}$$

$$V_2(h) = h \frac{\partial}{\partial x} - \frac{4}{3}y h_t \frac{\partial}{\partial U}$$

$$V_3(n) = -\frac{8}{3}n_t y \frac{\partial}{\partial x} + n \frac{\partial}{\partial y} + \left(-\frac{2}{3}x n_t + \frac{16}{9}y^2 n_{tt} \right) \frac{\partial}{\partial U}$$

$$V_4(s) = s \frac{\partial}{\partial U}$$

To our knowledge, the symmetry groups of eq. (2) have not been studied in literature.

Summary

We have presented the auxiliary equation method and Lie symmetry transformation to construct more general exact solutions of NLPDE with the aid of Maple. We have successfully obtained many new exact traveling wave solutions which may be useful for describing certain non-linear physical phenomena in fluid. It is shown that the algorithm can be also applied to other NLPDE in mathematical physics. Hence, the further study is needed.

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