AN APPROXIMATE ANALYTICAL (INTEGRAL-BALANCE) SOLUTION TO A NON-LINEAR HEAT DIFFUSION EQUATION

by

Jordan HRISTOV*

Department of Chemical Engineering, University of Chemical Technology and Metallurgy, Sofia, Bulgaria

Original scientific paper
DOI: 10.2298/TSCI140326074H

The paper presents a closed form approximate solution of the non-linear diffusion equation of a power-law non-linearity of the diffusivity developed by the heat-balance integral method. The main step in the initial transformation of the governing equation avoiding the Kirchhoff transformation is demonstrated. The consequent application of the integral method is exemplified by a solution of a Dirichlet problem with an approximate parabolic profile. Cases with predetermined positive integer and optimized non-integer exponents have been analyzed.

Key words: *non-linear diffusion, approximate integral solution, power-law diffusivity*

Introduction

The paper addresses the non-linear diffusion equation with a power-law dependent diffusivity $a = a_0 u^m$ [1] of integer positive index (exponent), namely:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left(a_0 u^m \frac{\partial u}{\partial x}\right), \quad m > 0, \quad u(0, t) = 1, \quad u(x, 0) = 0 \tag{1}$$

This model describes a variety of physical processes and in contrast to the linear diffusion equation (m = 0), eq. (1) is uniformly parabolic in any region where u is not zero, but degenerates in the vicinity of any point where u = 0 [2]. The main feature of this type of degeneracy is that any disturbances propagate at finite speed giving rise to a front or interface in the solution. Therefore, owing the non-linearity of the diffusivity coefficient there exist solutions with well-defined front separating the disturbed ($u \neq 0$) and the undisturbed medium [3, 4]. Fronts of this type are commonly observed in creeping flows [5, 6], non-linear heat conductivity [7, 8], diffusion with a concentration-dependent diffusivity coefficient [2, 9], *etc.*

The range of processes described by (1) is wide. The second-order equations with m = 2 is known as the porous media equation modelling of gas filtration in porous media [10-12]. Models (m = 3) are relevant to the process of isolation oxidation of silicon [13] and the lubrication theory approximation [5]. With m = 1 we have the Boussinesq equation [14] or a non-linear reaction-diffusion equation [15]. Many problems with various positive integer values of m are analyzed in [3, 6, 16] and the references therein.

* Corresponding author; e-mail: jordan.hristov@mail.bg

The difficulties inherent in obtaining solutions for this class of equations have motivated a variety of solution methods, both exact and approximate ones. There exist several approaches to solve eq. (1), among them:

– waiting-time solutions [6, 14, 17] describing evolution of $u(x)$ behind a front at a fixed position during a finite *waiting time* t_w,
– asymptotic methods [11, 13],
– similarity solutions [1, 4, 18, 19] using the Boltzmann similarity variable,
– analytic methods, based on the moment method, about solutions close the front [2], and
– the Kirchhoff transformation [20] $w=\int_0^u u^m \mathrm{d}u$ is the common approach to transform eq. (1) into:

$$\frac{\partial w}{\partial t}=a_0\frac{\partial^2 w}{\partial x^2} \tag{2}$$

The final solution may be developed either analytically [8, 13, 21-24]. For accuracy of the literature background, Heat-balance integral method (HBIM) to heat conduction with temperature-dependent diffusivity has been applied by Goodman [26] by a quasi-Kirchhoff transformation involving only the thermal properties at the surface = 0.

Approximate solution

In this paper we focus on an approximate solution of the Dirichlet problem by the heat-balance integral method [26] and a generalized parabolic profile [27, 28]. The approach avoids the Kirchhoff transform and by change of the variables $\varphi=u^m$ and $\tau = t/m$ allows eq. (1) to be expressed as:

$$\frac{\partial\varphi}{\partial\tau}=a_0\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+m\varphi\frac{\partial^2\varphi}{\partial x^2}\right],\quad \varphi(0,t)=1,\quad \varphi(x,0)=0 \tag{3}$$

The structure of eq. (3) reveals that the time evolution of φ is a result of superposition of non-linear wave propagation (the first term in RHS) and a diffusion (the second term in RHS) [12].

The approximate solution to eq. (3), with a Dirichlet boundary condition ($u = u_s = 1$, $x = 0$, $t = 0$, and $u = 0$, $x \to \infty$, $t > 0$), is expressed by a parabolic profile with undefined exponent $\varphi_a = (1 - x/\delta)^q$ [27] behind the front $\delta(t)$ and Goodman's boundary conditions $\varphi=0, \partial\varphi/\partial x=0$ for $x = \delta$ [26]:

$$\varphi_a=\left(1-\frac{x}{\delta}\right)^q \Rightarrow u_a=\left(1-\frac{x}{\delta}\right)^{\frac{q}{m}} \tag{4a, b}$$

where $u_a=(u-u_\infty)/(u_s-u_\infty)$ and $\varphi_a=(\varphi-\varphi_\infty)/(\varphi_s-\varphi_\infty)$, , respectively.

The conditions of a sharp front at $\delta(t)$ are satisfied because for $x = \delta$ we have: $u=0$ and $\partial u/\partial x=0$ as well as $\varphi=0, \partial\varphi/\partial x=0$.

Integrating eq. (3) from 0 to δ and applying the Leibniz rule to the left-side, we have:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\delta \varphi\,\mathrm{d}x=a_0\left[\int_0^\delta\left(\frac{\partial\varphi}{\partial x}\right)^2\mathrm{d}x+\int_0^\delta m\varphi\frac{\partial^2\varphi}{\partial x^2}\mathrm{d}x\right] \tag{5}$$

The use of $\varphi_a(x)$ in eq. (5) instead $\varphi(x)$ results in an ordinary differential eq. (6a) defining the propagation of the front $\delta(t)$. Taking into account that $\delta(t = 0) = 0$ and $\tau = t/m$ we have eq. (6b):

$$\frac{\mathrm{d}\delta^2}{\mathrm{d}\tau} = a_0 \frac{2q(q+1)}{2q-1}\left(\frac{q}{m} + q - 1\right) \Rightarrow \delta = \sqrt{a_0 t}\sqrt{2q(q+1)}\,\Delta_m \tag{6a, b}$$

In the expression (6b) the term $(a_0 t)^{-1/2}$ $[2q(q + 1)]^{-1/2} = \delta_0$ is the front depth in case of $m = 0$ [27]. The ratio $\Delta_m = \delta/\delta_0$ shows that the penetration depth decreases with increase in the non-linearity of the model (increase in the value of m), namely:

$$\Delta_m = \sqrt{\frac{1}{m(2q-1)}\left[q\left(\frac{m+1}{m}\right) - 1\right]} \tag{7}$$

The approximate profile can be expressed as {denoting $\eta = x/[(a_0 t)^{-1/2}]$ and $F_a = [2q(q + 1)]^{1/2}$} as:

$$\varphi_a = \left(1 - \frac{\eta}{F_q \Delta_m}\right)^q \Rightarrow u_a = \left(1 - \frac{\eta}{F_q \Delta_m}\right)^{\frac{q}{m}} \tag{8a, b}$$

Constraints imposed on the exponent of the approximate profile

Approach 1

First of all, we have two physically imposed conditions: (1) From the conditions imposed to eq. (1) we have $m > 0$. (2) The front depth should be positive $\delta > 0$ that implies $[1/(2q - 1)](q/m + q - 1) > 0$, and we have two cases: (1) $q > 1/2$ and $q > m/(m + 1)$ if both terms are assumed positive, and (2) $q > 1/2$ and $q < m/(m + 1)$ if the both terms are assumed as negative.

If *we suggest integer order* of the approximate profile (not a linear one with $q = 1$) of the profile approximating the solution of the transformed eq. (3), then the first conditions are reasonable. The constraints are automatically satisfied if this approximate profile is defined as quadratic ($q = 2$) or cubic ($q = 3$) as in the classical HBIM [26, 27]. In this case, if $m = 1$, then $q > 1/2$, which automatically satisfies the condition $\delta > 0$. Further, with $m = 3$ we should defined $q > 1/3$ and $q > 0.75$, respectively.

Approach 2

The constraints applied to the exponent q established by Approach 1 are mechanistic ones, *i. e.* imposed by the final form of the approximate profile and the initial assumption that q should be integer. Now, we focus the attention on the fact that the approximate profile satisfies the heat-balance integral (5) but not the original heat conduction eq. (1), a detail omitted in the Approach 1.Therefore, the function $\sigma[u_a(x, t)]$:

$$\sigma[u_a(x,t)] = \frac{\partial u_a}{\partial t} - \frac{\partial}{\partial x}\left(a_0 u_a^m \frac{\partial u_a}{\partial x}\right) \tag{9}$$

should be zero if u_a matches the exact solution.

With the approximate profile (8b), denoting $n = q/m$ we have $u_a = (1 - x/\delta)^n$ and the next goal is to attain a minimum of $\sigma[u_a(x, t)]$ for a certain value of the exponent n (the only unspecified parameter of the approximate profile). In this case we have:

$$\frac{\partial u_a}{\partial t} = \frac{nx}{\delta^2}\frac{\mathrm{d}\delta}{\mathrm{d}t}\left(1-\frac{x}{\delta}\right)^{n-1}, \quad \frac{\partial}{\mathrm{d}x}\left(u_a^m \frac{\partial u_a}{\partial x}\right) = \frac{n[n(m+1)-1]}{\delta^2}\left(1-\frac{x}{\delta}\right)^{n(m+1)-2} \tag{10a,b}$$

Then, for example at $x = 0$:

$$\sigma_T(0,t) = -\frac{n[n(m+1)-1]}{\delta^2} \tag{11}$$

Searching for positive values of n, the heat equation is satisfied for $n = 1/(m + 1)$. However, in order to satisfy the Goodman's boundary conditions $u_a(\delta, t) = u_a(\delta, t)/\partial x = 0$, it is required that $n(m + 1) > 1$, that is $n > 1/(m + 1)$. Further, for $x \to \delta$ we have:

$$\sigma_T(\delta,t) = \lim_{x\to\delta}\sigma_T(\delta,t) = -\frac{n[n(m+1)-1]}{\delta^2}\lim_{x\to\delta}\left(1-\frac{x}{\delta}\right)^{n(m+1)-2} \tag{12}$$

With the previous constraint, $n > 1/(m + 1)$, it follows from eq. (12) that the heat conduction equation is satisfied at $x = \delta$ when $n = 2/(m + 1)$. For $m = 0$, we have $n > 2$ as it was established by Mitchell and Myers [28] (see further in this article).

Error of approximation and optimal exponents

Langford criterion: general approach and integer order of the exponent q

Following the Langford criterion [29] the accuracy of the approximations can be quantified by calculating the mean-squared error of approximation, namely:

$$E_u = \int_0^{\delta}\left[\frac{\partial u_a}{\partial t} - \frac{\partial u_a}{\partial x}\left(a_0 u_a^m \frac{\partial u_a}{\partial x}\right)\right]^2 \mathrm{d}x =$$

$$= E_\varphi = \int_0^{\tau}\left[\frac{\partial \varphi_a}{\partial \tau} - a_0\left(\frac{\partial \varphi_a}{\partial x}\right)^2 - a_0 m\varphi\frac{\partial^2 \varphi_a}{\partial x^2}\right]^2 \mathrm{d}x \to \min \tag{13a,b}$$

For simplicity of calculations we will use the form of (13b). Taking into account the expressions for $\delta(\tau)$, as well as eqs. (6b) and (7), we have (avoiding the cumbersome calculations):

$$E_\varphi = \frac{1}{\delta^3}\left\{\frac{\mathrm{d}\delta}{\mathrm{d}\tau}\frac{3q^2-4q-1}{(2q-1)(3q-2)}F_q\Delta m + a_0 mq(q-1)\left[\frac{3q-3}{3q-2} + a_0 mq(q-1)\right]\right\} \to \min \tag{14}$$

From eq. (14) we have that E_φ decays in time with a speed $\delta^{-3} \equiv \tau^{-3/2}$. Moreover, $\mathrm{d}\delta/\mathrm{d}\tau = a_0/2\tau^{1/2}$, so neglecting the terms decaying in time, *i. e.* the method of Myers [30] we reduce eq. (14) to:

$$a_0 mq(q-1)\left[\frac{3q-3}{3q-2} + a_0 mq(q-1)\right] \to \min \tag{15}$$

Setting expression (15) equal to zero, the trivial solution $q_1 = 0$ is unphysical, while the second one is $q_2 = 1$. Hence, from expression (15) we have (with assumption $a_0 = 1$ for simplicity) an extreme case:

$$q^2 - q + \frac{3}{2m} = 0 \Rightarrow q = -\left(\frac{1}{2} + \frac{\sqrt{m^2 - 6m}}{2m}\right) \qquad (16a, b)$$

Further, from eq. (16a) we have the conditions: $q > 0 \Rightarrow m > 0$, and $m^2 - 6\,m > 0 \Rightarrow$ $\Rightarrow m > 6$.

Calculating the errors of approximation for particular values of m and n, we have to bear in mind that the common values of $a_0 \sim 10^{-6}$ m^2/s. Hence, the second term in expression (15) can be neglected (order of $O(10^{-12})$ which results in:

$$E_\varphi \to a_0\left[mq(q-1)\frac{3q-3}{3q-2}\right] = a_0 e_\varphi(m, q) \qquad (17)$$

The classic application of the heat-balance integral method assumes integer values of the exponent q and some calculated values of $e_\varphi(m, q)$, as well as the factors F_q and Δ_m, are summarized in tab. 1. The data reveal that we have a minimal error of approximation with $e_\varphi(m, q) = e_\varphi(2, 2) \equiv 3a_0$. The calculated values of q/m strongly indicate *that the exponent* $n = q/m$ *of the final approximate profile is not intege*r even though we used $q = 2$ and $q = 3$. The values of Δ_m for all case with $q > 1$ are $\Delta_m < 1$ which simply means that *the front of penetration becomes shorter with increase in the diffusion non-linearity*, *i. e*. increase in the value of m [7], a fact well-known where the diffusion processes with a power-law diffusivity [31] which are classified as: *slow diffusion* with $m > 0$ and fast diffusion with $m < 0$.

Table 1. Correction factors of the penetration depth length and estimation of the mean squares errors of approximation

	$m = 0$		$m = 1$		$m = 2$		$m = 3$	
	Δ_m	Δ_m	F_a	Δ_m	F_a	Δ_m	F_a	Δ_m
$q = 2$	–	–	3.464	1	3.464	0.577	3.464	0.192
q/m	–	–	2	–	1	–	0.666	–
$L_2 \equiv e_\varphi(m, q)$	–	–	–	–	$3\,D_0$		$4.5\,D_0$	
$q = 3$	–	–	4.899	–	4.899	–	4.899	–
q/m	–	–	3	1	1.5	0.591	1	0.447
$L_2 \equiv e_\varphi(m, q)$	–	–	–	–	$(72/7)\,D_0$		$(108/6)\,D_0$	

The numerical experiments with fixed integer values of q presented in figs. 1(a) and (b) clearly reveal that with increase in the value of the exponent m there exists a retardation in the propagation of the front $\delta(t)$. This is well presented by the line A in the plane $m - \eta$. Further, there is a decreases about 3 times in the penetration depth length when the value of m increases from 0.6 to about 2.75, fig. 1(a). Moreover, there is a change in the shape of the profile from concave to convex with increase in the value of m, a fact which we will comment further in this article. These numerical experiments present qualitative results only because both q and m are predetermined.

Modified method of Mitchell and Myers [28]

Now, we try to find answers to the question raised in the last paragraph of the preceding section: what is the optimal ratio q/m? In this direction, we refer to the approach Mitchell

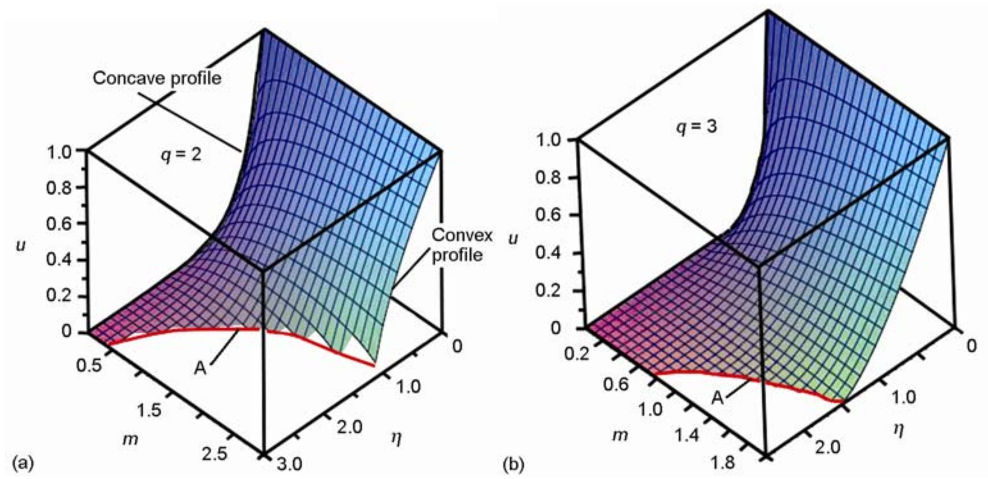

Figure 1. Approximate profiles as functions of the similarity variable *η* and *m* > 1 for two predetermined integer exponents of the profile exponent: *q* = 2, and *q* = 3

and Myers [28] representing the approximation profile $V = (1 - x/\delta)^q$ (see 4a) in a new co-ordinate $\xi = x/\delta$, $0 < \xi < 1$, $V(\xi, t) = (1 - \xi)^q$. Hence, the heat eq. (3) can be expressed as:

$$\frac{\partial V}{\partial t} - \frac{\mathrm{d}\delta}{\mathrm{d}t}\frac{\xi}{\delta}\frac{\partial V}{\partial \xi} = a_0\left[\frac{m}{\delta^2}\left(\frac{\partial V}{\partial \xi}\right)^2 + \frac{V^m}{\delta^2}\frac{\partial^2 V}{\partial \xi^2}\right] \tag{18}$$

Then, setting $\partial V/\partial t = 0$ which comes naturally from the definition of V and the transfer from a moving region $0 \le x \le \delta$ to the fixed one $0 \le \xi \le 1$, as well as with $a_0 = 1$ (for convenience) the equivalent of the squared error function is:

$$E_{MT} = \int_0^1\left[\delta\frac{\mathrm{d}\delta}{\mathrm{d}t}\xi\frac{\partial V}{\partial \xi} + m\left(\frac{\partial V}{\partial \xi}\right)^2 - V^m\frac{\partial^2 V}{\partial \xi^2}\right]^2 \mathrm{d}\xi \tag{19}$$

The method developed in [28] uses the fact that for $m = 0$ the product $\delta(\mathrm{d}\delta/\mathrm{d}t)$ is time-independent and the function E_{MT} depends only on q. For the Dirichlet problem analyzed here this specific feature is also valid (for any value of m) because the non-linearity has no effect on the time in the penetration depth and the squared-root still exists, eq. 6(b), as in the linear diffusion: the product is $\delta(\mathrm{d}\delta/\mathrm{d}t) = |a_0\,(F_q\Delta_m)^2|/2$.

Integrating eq. (19) and using the HBIM solutions of eqs. (6b) and (8b) we may express E_{MT} as $E_{MT(\mathrm{HBI})} = e_{MT(\mathrm{HBI})}(q, m)/\mathrm{t}^2$ with $e_{MT(\mathrm{HBI})}(q, m)$ which is time-independent. The expression of $e_{MT(\mathrm{HBI})}(q, m = 1)$, for example, is:

$$E_{MT(\mathrm{HBI})} = \left(\frac{1}{t^2}\right)\cdot \left[\frac{1}{3}\frac{108 - 279q + 1891q^4 - 762q^2 + 1787q^3 - 3878q^8 + 1440q^8 + 1290q^7 - 1237q^6}{(3q-1)(3q-2)(q-1)(4q-3)(2q+3)(4q^2-1)}\right] \tag{20}$$

Therefore, the error of approximation decreases in time with a rate t^2 and we have to minimize $E_{MT(\mathrm{HBI})}$ with respect to q. To this end, we have two options: (1) to minimize

$E_{MT(\mathrm{HBI})}$ with respect to q at given m in the zone for $q > 1$ (precisely for $q > 2$ to avoid the singularity near $q = 1$) where the curve $E_{MT(\mathrm{HBI})}$ is decaying smoothly, fig. 2, and find the optimal exponents, and (2) to solve $E_{MT(\mathrm{HBI})}(q, m) = 0$ finding approximate roots and then to find for which of them $E_{MT(\mathrm{HBI})}$ obtains minima. The first approach seems reasonable because $E_{MT(\mathrm{HBI})}$ is a scaled function. With the second approach we formally envisage exact solutions, although, in fact, we look for approximate ones; performing numerical solutions of $E_{MT(\mathrm{HBI})}(q, m) = 0$ we really determine approximately points where $E_{MT(\mathrm{HBI})}$ gets minima, because practically exact solutions do not exist. Then, by evaluation of $E_{MT(\mathrm{HBI})}$ for these roots we may establish the optimal exponents of the profile.

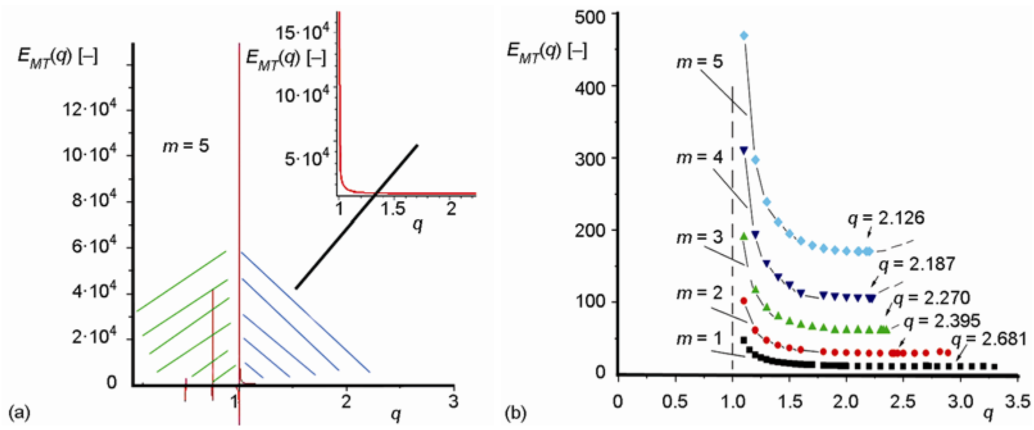

Figure 2. Behaviour of the function $E_{MT(\mathrm{HBI})} = f(q)$ for $m = 5$. (a) Overall behavior with zones: $q < 1$ and $q > 1$; (b) enlarged zone $1 < q < 3.5$ with minima determined by minimization of $E_{MT(\mathrm{HBI})}$

With the first approach, all values of q are generally greater than 1, tab. 2. Oppositely, with the second approach all values of q are lower than 1. For the second case we provide some details (the data are summarized in tab. 2).

Table 2. Optimal exponents of the parabolic profile developed by the method of Mitchell and Myers [28] and HBIM approximate solutions. Two approaches in determination of the optimal exponents

	Numerical solution of $E_{MT(\mathrm{HBI})}$, $n < 1$			By minimization of $E_{MT(\mathrm{HBI})}$, $n > 2$		
	q_{opt}	n_{opt}	$E_{MT(\mathrm{HBI})} = f(q)$	q_{opt}	n_{opt}	$E_{MT(\mathrm{HBI})} = f(q)$
$m = 0.5$	–	–	–	3.319	6.638	6.330
$m = 1$	0.610	0.610	0.02790	2.681	2.681	11.0683
$m = 2$	0.509	0.254	$7.47 \cdot 10^{-3}$	2.395	1.197	29.4347
$m = 3$	0.302	0.100	$2.22 \cdot 10^{-3}$	2.270	0.756	60.5617
$m = 4$	0.299	0.074	$7.524 \cdot 10^{-3}$	2.187	0.546	106.697
$m = 5$	0.256	0.051	$1.11 \cdot 10^{-3}$	2.126	0.425	170.220

For $m = 1$ the solution of $e_{MT(\mathrm{HBI})}(q, 1) = 0$ provides 3 roots (solved numerically by Maple): $q_1 \approx 0.2106$, $q_2 \approx 0.3794$, and $q_3 \approx 0.6109$. Taking into account the constraints $n = q/m >> 1/(m + 1)$ the only root satisfying it is $q_3 \approx 0.6109$ with $E_{MT(\mathrm{HBI})}(q, 1) \approx 0.02790$. Hence, from $n = q/m$ we have $n_{\mathrm{opt}} = 0.6109$. Similarly, for $m = 2$, the solution of $e_{MT(\mathrm{HBI})}$

$(q, 2) = 0$ provides 6 roots but only 3 of them satisfy the constraint, namely: $q_2 \approx 0.3907$ and $q_3 \approx 0.509$. The evaluation of the $E_{MT(\text{HBI})}(q, 2)$ for any of them provides a minimal error of approximation for $e_{MT(\text{HBI})}(q_3, 2) \approx 7.46 \cdot 10^{-3}$.

For $m = 3$, the optimization procedure results only one root satisfying the constraint $q \approx 0.302$ and $E_{MT(\text{HBI})}(q, 3) \approx 2.22 \cdot 10^{-3}$. Additionally, for $m = 4$ we have $q_{\text{opt}} \approx 0.2998$ with $E_{MT(\text{HBI})}(q, 4) \approx 7.524 \cdot 10^{-4}$, while for $m = 5$ the optimal exponent is $q_{\text{opt}} \approx 0.2567$ with $E_{MT(\text{HBI})}(q, 5) \approx 1.11 \cdot 10^{-3}$.

Now, let us look again at the data summarized in tab. 1. The ratios $q/m = n$ are non-integer but they are far away from the optimal ones: for $m = 1$, we have $n_{opt} \approx 0.6109$, while $q/m = 2$ for $q = 2$, and $q/m = 3$ for $q = 3$ are defined *ad hoc* trough predetermined integer values of q: the values of the error measure are $E_{MT(\text{HBI})}(q = 2,1) \approx 2.9621$ and $E_{MT(\text{HBI})}(q = 3,1) \approx 7.5623$, respectively. In this context, for $m = 2$, the ratios $q/m = 1$ and $q/m = 1.5$ lead to $E_{MT(\text{HBI})}(q = 1, m = 2) \approx 0.533$ and $E_{MT(\text{HBI})}(q = 3, m = 2) \approx 1.558$, respectively. It is evident, that, to some extent, these cases may be solved approximately with integer values of q and acceptable errors of approximations, but the approach to determine the optimal exponent (either q or $n = q/m$) is the accurate one.

The plots of the approximate solutions generated by the parabolic profile with optimal exponents determined by the two approaches are shown in figs. 3(a) and (b).

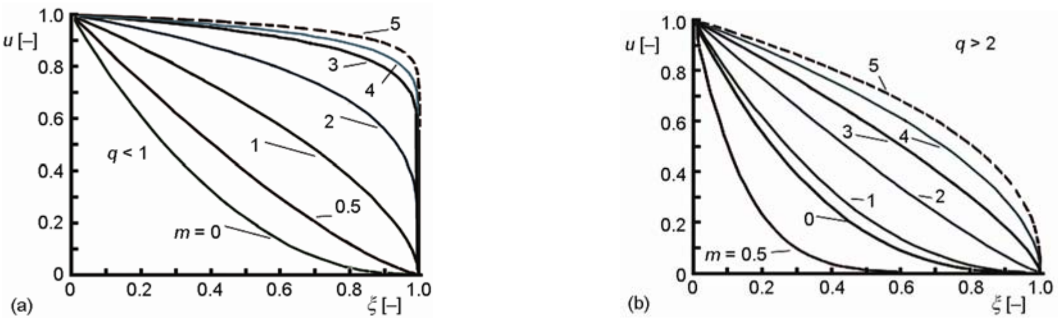

Figure 3. Dimensionless temperature profiles with various degrees of non-linearity (the parameter *m*); (a) Approximate solutions with n_{opt} determined by numerical solution of $E_{MT(\text{HBI})} = 0$, and $q < 1$, (b) Approximate solutions with n_{opt} determined by minimization of $E_{MT(\text{HBI})}$ in the range $2 < q < 3.5$

The plots in fig. 3(a) reveal strong change in the profile shape from concave to convex when the optimal exponent is determined by $q < 1$, while the same behaviour exhibited by the profiles in fig. 3(b) is not so well demonstrated. Therefore, there exist ambiguous results and the situation should be clarified by comparing the approximate HBIM solution to reference solutions of the problem. The determination of the correct exponents is the principle question and to find the right answer we will compare our results to the series solution of Heaslet and Alksne [32] as it was done in other studies, in [2] for instance.

Determination of the correct exponent of the profile and numerical examples

The comparison of the approximate HBIM solutions to the series solution of Heaslet and Alksne [32] presented in figs. 4 and 5, definitely indicates that the concave profiles developed on the basis of optimal $q > 2$, fig. 5(a), are by far away from the series solutions, while the convex profiles developed with $q < 1$, fig. 4(a), are too close to them. The profiles in fig. 4(a) reveal that the HBIM solutions are more adequate (close to the series solutions) with increase in the value of m. Further, the pointwise errors between the HBIM and the series solutions support this standpoint, that is, the profiles generated on the basis of $q < 1$ demonstrate pointwise errors less than 4% in contrast to 25-30% when the optimal exponents are determined on the basis of $q > 2$.

The plots in figs. 3, 4, and 5 are presented in the form $u(\xi, t) = (1 - \xi)^n$, where $0 < \xi = x/\delta < 1$. It is worth to note, that the value of δ is different for different values of the parameter m because δ depends on n and m. Alternatively, ξ may be presented as $\xi = \eta/F_q\Delta_m$,

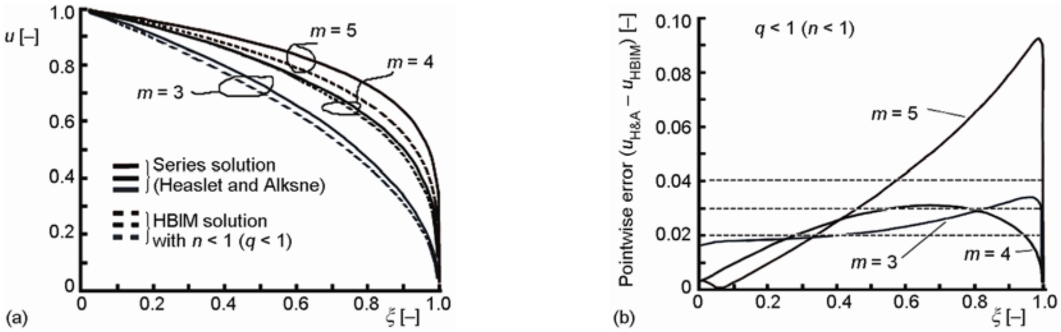

Figure 4. Dimensionless temperature profiles determined on the basis of $q < 1$; (a) comparison to the series solution of Heaslet and Alksne [32] (4 terms solutions), (b) pointwise error between the approximate HBIM solutions and the series solutions [32]

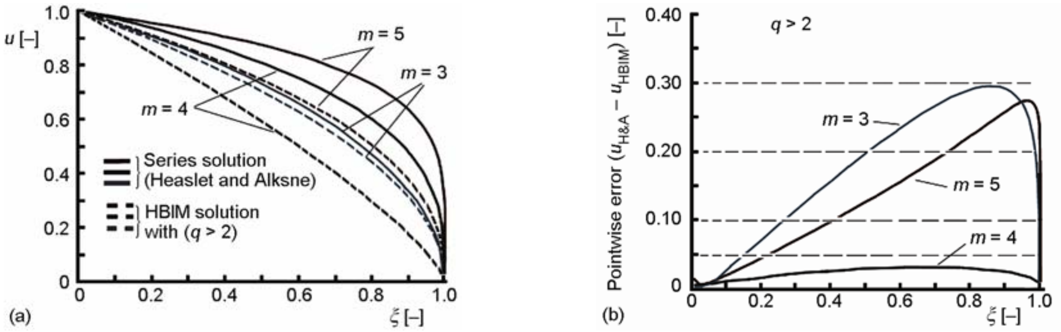

Figure 5. Dimensionless temperature profiles determined on the basis of $q > 2$; (a) comparison to the series solution of Heaslet and Alksne [32] (4 terms solutions), (b) pointwise error between the approximate HBIM solutions and the series solutions [32]

eqs. 8(a, b). When the profiles are expressed against the similarity variable η only, then the curves cross the abscissa at different positions because the condition $u = 0$ means $\eta = F_q\Delta_m$ which depends on both the values of m and n_{opt} as it is shown in fig. 6. Otherwise, when $\xi = \eta/F_q\Delta_m$ as independent variable, all curves cross the abscissa at $\xi = 1$; this allows comparing the approximate HBIM profiles and those developed by the series solutions, figs. 4 and 5.

The plots in fig. 6 clearly show the retardation effect of the non-linearity with increase of the parameter m. The increase in m reduces the penetration depth and this the effect is visible when the similarity variable η is used as independent variable (fig. 6), but becomes indistinguishable when the profiles are presented against $\xi = x/\delta = \eta/F_q\Delta_m$ as independent variable, figs. 3, 4(a), and 5(a).

As a final comment, we have to mention the parabolic profile used in the HBIM solutions are generates convex distributions with steep fronts only when $n < 1$, otherwise for $n > 1$ the parabolic profile generates concave distributions.

Conclusions

The paper reports an approximate solution of non-linear heat conduction problem with power-law heat diffusivity (positive exponent) and Dirichlet boundary condition

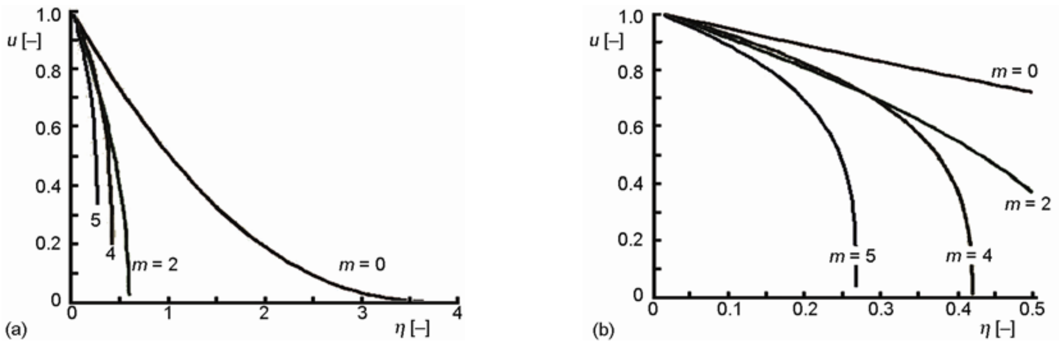

Figure 6. Dimensionless temperature profiles with various degrees of non-linearity (*m* as a parameter); (a) large-scale profiles, (b) short-distance profiles. Note: n_{opt} is determined by numerical solution of $E_{MT(HBI)} = 0$, *i. e.* on the basis of $q < 1$, tab. 2

by the heat-balance integral method (HBIM) utilizing a parabolic profile with undefined exponent. The application of the HBIM is possible due to an initial transformation of the non-linear degenerate diffusion equation into an equivalent diffusion-wave equation, thus avoiding the common conjectures such as the Kirchhoff transformation.

The approximate solution allows to be optimized with respect to the value of the exponent of the parabolic profile. The application the global optimization through the method of Mitchell and Myers [28] provides optimal non-integer exponents through a minimization of the squared-error function with respect to q, then defining $n = q/m$. The numerical simulations clearly indicate the retardation of the propagation of the heat front as the exponent m of the power-law diffusivity increases.

The numerical experiments demonstrate definitively that convex profiles of the correct HBIM solution have non-integer exponents lower than 1; these profiles approach the series solutions of Heaslet and Alksne [32] and the maximal absolute error between them does not exceed 0.1 for $m = 5$.

References

[1] Hill, J. M., Similarity Solutions for Non-Linear Diffusion – a New Integration Procedure, *J. Eng, Math*, *23* (1989), 1, pp. 141-155

[2] Prasad, S. N., Salomon, J. B., A New Method for Analytical Solution of a Degenerate Diffusion Equation, *Adv. Water Research, 28* (2005), 10, pp. 1091-1101

[3] Smyth, N. F. *et al*., High-Order Non-linear Diffusion, *IMA Journal of Applied Mathematics, 40* (1988), 1, pp. 73-86

[4] Zel'dovich, Ya, B., Kompaneets, A. S., On the Theory of Heat Propagation for Temperature Dependent Thermal Conductivity, in: *Collection Commemorating the 70th Anniversary of A. F. Joffe, Izvestia Akad. Nauk SSSR,* 1950, pp. 61-71

[5] Buckmaster, J., Viscous Sheets Advancing over Dry Beds, *J. Fluid Mech., 81* (1977), 4, pp. 735-756

[6] Marino, B. M., *et al*., Waiting-Time Solutions of a non-Linear Diffusion Equation: Experimental Study of a Creeping Flow Near a Waiting Front, *Physical Review E*, *54* (1996), 3, pp. 2628-2636

[7] Lonngren, K. E., *et al*., Field Penetration into Plasma with non-Linear Conductivity, *The Physics of Fluids*, *17* (1974), 10, pp. 19191-1920

[8] Khan, Z. H., *et al*., Effect of Variable Thermal Conductivity on Heat Transfer from a Hollow Sphere with Heat Generation Using Homotopy Perturbation Method, *Proceedings,* ASME Summer Heat Transfer Conference, Jacksonville, Fla., USA 2008, Article HT2008-56448
[9] Pattle, R. E. Diffusion from an Instantaneous Point Source with Concentration-Dependent Coefficient, *Quart. J. Mech. Appl. Math*., *12* (1959), 4, pp. 407-409
[10] Muskat, M., *The Flow of Homogeneous Fluids through Porous Media*, McGraw-Hill, New York, USA, 1937
[11] Peletier, L. A., Asymptotic Behavior of Solutions of the Porous Media Equations, *SIAM J. Appl. Math, 21* (1971), 4, pp. 542-551
[12] Aronson, D. G., *The Porous Medium Equation*, in: *Non-linear Diffusion Problems, Lecture Notes in Math. 1224* (Eds. A. Fasano, M. Primicerio), Springer, Berlin, 1986, pp. 1-46
[13] King, J. R., The Isolation Oxidation of Silicon: The Reaction-Controlled Case, *SIAM J. Appl. Math., 49* (1989), 4, pp. 1064-1080
[14] Lacey, A. A., *et al*., Waiting-Time Solutions of a non-Linear Diffusion Equation. *SIAM J. appl. Math., 42* (1982), 6, pp. 1252-1264
[15] Atkinson, C., *et al*., Traveling Wave Solutions for some non-Linear Diffusion Equations, *SIAM J. Appl. Math*., *12* (1981), 6, pp. 880-892
[16] Strunin, D. V., Attractors in Confined Source Problems for Coupled non-Linear Diffusion, *SIAM J. Appl. Math*., *67* (2007), 6, pp. 1654-1674
[17] Nasseri, M., *et al*., New Analytical Solution to Water Content Simulation in Porous Media, *J. Irrigation and Drainage Eng., 138* (2012), 4, pp. 328-335
[18] Hill, D. L., Hill, J. M., Similarity Solutions for non-Linear Diffusion-Further Exact Solutions, *J. Eng, Math*, *24* (1990), 1, pp. 109-124
[19] Hill, J. M., Hill, D. L., On the Derivation of First Integrals for Similarity Solutions, *J. Eng., Math.*, *25* (1991), 2, pp. 287-299
[20] Tomatis, D., Heat Conduction in Nuclear Fuel by the Kirchhoff Transformation, *Ann. Nucl. Energy., 57* (2013), July, pp. 100-105
[21] Knight, J. H., Philip, J. R., Exact Solutions in non-Linear Diffusion, *J. Eng. Math*., *8* (1974), 3, pp. 219-227
[22] Kolpakov, V. A., *et al*., Determination of the Surface Temperature of Sample in Region of its Interaction with a Nonelectrode Plasma Flow Using the Kirchhoff Transformation of a Quadratic Function, *Technical Physics*, *58* (2013), 11, pp. 1554-1557
[23] King, J. R., Integral Results for non-Linear Diffusion Equations, *J. Eng. Math*., *25* (1991), 2, pp. 191-205
[24] Paripour, M., *et al*., Analytic Solutions to Diffusion Equations, *Math. Comp. Model*., *51* (2010), 5-6, pp. 649-657
[25] Walker, G. G., Scott, E. P., Evaluation of Estimation Methods for High Unsteady Heat Fluxes from Surface Measurements, *J. Thermophys. Heat Transfer*, *12* (1998), 4, pp. 543-551
[26] Goodman T. R., Application of Integral Methods to Transient Non-linear Heat Transfer, *Advances in Heat Transfer* (Eds. T. F. Irvine, J. P. Hartnett, 1 (1964), Academic Press, San Diego, Cal., USA, pp. 51-122
[27] Hristov, J. The Heat-Balance Integral Method by a Parabolic Profile with Unspecified Exponent: Analysis and Benchmark Exercises, *Thermal Science, 13* (2009), 2, pp. 27-48
[28] Mitchell, S. L., Myers, T. G., Application of Standard and Refined Heat Balance Integral Methods to One-Dimensional Stefan Problems, *SIAM Review, 52* (2010), 1, pp. 57-86
[29] Langford, D., The Heat Balance Integral Method, *Int. J. Heat Mass Transfer, 16* (1973), 12, pp. 2424-2428
[30] Myers, J. G., Optimizing the Exponent in the Heat Balance and Refined Integral Methods, *Int. Comm. Heat Mass Transfer, 36* (2009), 2, pp. 143-147
[31] Sadighi, A., Ganji, D. D., Exact Solutions of non-Linear Diffusion Equations by Variational Iteration Method, *Comp. Math Appl*., *54* (2007), 7-8, pp. 1112-1121
[32] Heaslet, M. A., Alksne, A. Diffusion from a Fixed Surface with a Concentration-Dependent Coefficient, *J. Soc. Ind. Appl. Math.*, *9* (1961), 4, pp. 463-479

Paper submitted: March 3, 2012
Paper revised: April 19, 2012
Paper accepted: Jun 13, 2012