TURBULENCE TRANSITION THRESHOLD OF A BUOYANCY-DRIVEN FLOW OVER AN ISOTHERMAL VERTICAL WALL

by

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The integral formalism has been developed in an attempt to predict the mathematical transition threshold from laminar to turbulent regimes in a free convection boundary-layer flow adjacent to an isothermal vertical wall and also to gain a better understanding in the prediction of the turbulence mechanism. The transition threshold is approached herein as the intersection of two fully developed laminar and turbulent flow regimes. The transition for the isothermal wall condition is the Prandtl number dependent and occurs differently depending on whether the dynamical or thermal viewpoints are considered. A change in the transition threshold behaviour is observed at Pr = 20.

Key words: *free convection, vertical wall, laminar-turbulent transition, transition threshold*

Introduction

The purpose of this note is to enhance the discussion on laminar to turbulent transition threshold in boundary-layer free convection on an isothermal vertical plate. The theoretical approach used in this paper is similar to that proposed by Varga *et al*. [1] in the uniformly heated vertical plate case and successfully extended to Newtonian nanofluids [2]. Identically to recent works [3] and for lack of physical criterion to define the transitional region, the transition threshold is regarded from a mathematical viewpoint as the point of intersection of both laminar and turbulent free convection regimes. Because little information is available in literature concerning the effect of the Prandtl number on the transition characteristics, a fully developed turbulent model is derived for a wide Prandtl number range and compared to that of the laminar case proposed and developed in details in Mladin *et al.* [4]. To formulate the mathematical modelling foundation and contrary to Eckert's theory commonly adopted in literature [5, 6] the integral formalism is derived here from the assumption that distinctive scaling lengths are considered for the dynamical and thermal boundary layer thicknesses and that the ratio $\Delta = \delta_T/\delta$ of the thermal

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boundary layer thickness (δ_T) to that of the dynamical one (δ) is only Prandtl number dependent [7] and not flow regime dependent. This concept, initially adopted in the laminar regime for both constant wall temperature [4] and constant wall heat flux density [8] thermal boundary conditions has been extended to the turbulent regime for a uniform heat flux condition [1]. Validations and deduced predictions for both laminar and turbulent regimes were shown to agree fairly with available experimental and numerical results in literature.

Turbulent free convection regime

The physical system consists of a vertical heated wall of constant temperature (isothermal). Heat is transferred at the wall surface by free convection mechanisms only. Assuming the Boussinesq's approximation and that the fully turbulent region starts from the leading edge of the plate, the time-averaged boundary layer equations for the conservation of mass, momentum and energy taking into account the turbulent eddy diffusivity of momentum and heat are written as:

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0 \tag{1}$$

$$U\frac{\partial \Theta}{\partial x}+V\frac{\partial U}{\partial y}=g\beta\Theta_{\mathrm{w}}+\frac{\partial}{\partial y}\left[(\nu+\nu_{\mathrm{t}})\frac{\partial U}{\partial y}\right] \tag{2}$$

$$U\frac{\partial \Theta}{\partial x}+V\frac{\partial \Theta}{\partial y}=\frac{\partial}{\partial y}\left[(a+a_{\mathrm{t}})\frac{\partial \Theta}{\partial y}\right] \tag{3}$$

where ν_t and a_t denote the eddy diffusivity of both momentum and heat, respectively.

In the set of turbulent boundary-layer equations (1)-(3) the empirical shearing stress relation and heat flux across the thermal boundary layer with the Colburn analogy [5] can be introduced as:

$$\frac{\tau_{\mathrm{w}}}{\rho}=(\nu+\nu_{\mathrm{t}})\left(\frac{\partial U}{\partial y}\right)_{\mathrm{y}=0}=0.0225u_x^2\sqrt[4]{\frac{\nu}{u_x\delta}} \tag{4}$$

$$\frac{q}{\rho C_p}=-(a+a_{\mathrm{t}})\left(\frac{\partial \Theta}{\partial y}\right)_{y=0}=\frac{\tau_{\mathrm{w}}\Theta_{\mathrm{w}}}{\rho u_{\mathrm{x}}\sqrt[3]{\mathrm{Pr}^2}} \tag{5}$$

where Θ_w is the wall temperature gradient assumed to be constant. Choosing the same velocity and temperature profiles as those first proposed by Eckert and Jackson [9], and commonly used since then, integrating the momentum and energy equations with respect to the $\varDelta \leq 1$ condition Mladin *et al*. [4] gives, after calculations, the following dynamical and thermal solutions of the turbulent free convection problem in terms of the dimensionless boundary layer thickness δ/x and local Nusselt number Nu_x:

$$\left.\frac{\delta}{x}\right|_{\mathrm{turb}}=0.04(\mathrm{Ra}_x)^{-1/10}\left[\frac{\mathrm{Pr}^{-13/3}}{\varDelta(\Pi_\varDelta)^8}(1+13.493(\Pi_\varDelta)\sqrt[3]{\mathrm{Pr}^2}\right]^{1/10} \tag{6}$$

$$\mathrm{Nu}_x\big|_{\mathrm{turb}}=0.057\sqrt[5]{(\mathrm{Ra}_x)^2}\left[\frac{\mathrm{Pr}^{-1/6}}{\sqrt[4]{\Pi_\varDelta}}(1+13.493(\Pi_\varDelta)\sqrt[3]{\mathrm{Pr}^2})\right]^{-2/5} \tag{7}$$

Π_Δ denotes the same Δ function as given in Varga *et al*. [1] and is defined as:

$$\Pi_\Delta = \sqrt[7]{\Delta^8}\left(\frac{7}{72} - \frac{7}{60}\Delta + \frac{21}{253}\Delta^2 - \frac{14}{435}\Delta^3 + \frac{7}{1332}\Delta^4\right) \tag{8}$$

Corresponding numerical values of Δ and Π_Δ are reported in tab. 1.

Laminar free convection regime

The reasoning associated with such an integral approach in the laminar case has been previously presented in details in [4]. Therefore, for the sake of brevity, only the solutions for the boundary layer thickness and the local Nusselt number are presented as:

$$\left.\frac{\delta}{x}\right|_{\text{lam}} = \frac{1}{\sqrt[4]{\text{Ra}_x}}\sqrt[4]{\frac{18144}{5}\text{Pr}\left(\frac{3}{5}\Delta - \frac{1}{3}\right)} \tag{9}$$

$$\left.\text{Nu}_x\right|_{\text{lam}} = \frac{2x}{\Delta\delta} \tag{10}$$

Laminar to turbulence transition threshold

Contrary to many analyses in literature where only heat transfer characteristics are interested, the present study aims at focusing both on dynamical and on thermal events in the transitional region. Therefore, fig. 1 shows the variation of both Nusselt numbers and the velocity layer thickness against the Rayleigh number Ra_x.

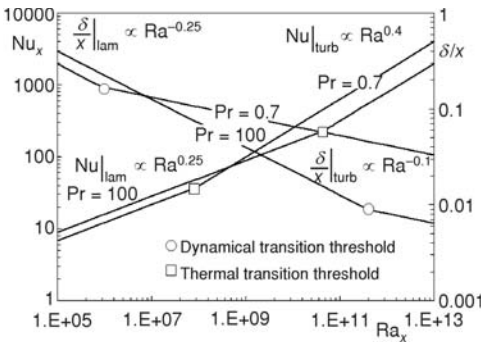

Figure 1. Local Nusselt number and velocity layer thickness *vs.* the Rayleigh number for Pr = 0.7 and Pr = 100

From fig.1 the mathematical transition threshold is clearly evidenced as the location where laminar and turbulent parameters interact together due to a change in slope in their evolution. It also appears that as well the Prandtl number as the thermal or dynamical study viewpoints are major parameters in the transition threshold. Both mathematical dynamical $(\text{Ra}_x)_\text{D}$ and thermal $(\text{Ra}_x)_\text{T}$ transitions are easily deduced analytically and given as:

$$(\text{Ra}_x)_\text{D} = 1.961\cdot10^{13}\sqrt[3]{\frac{\text{Pr}^{41/3}\,\Delta^2(\Pi_\Delta)^{16}(9\Delta-5)^5}{[1+13.493(\Pi_\Delta)\sqrt[3]{\text{Pr}^2}]^2}} \tag{11}$$

$$(\text{Ra}_x)_\text{T} = 2.129\cdot10^{6}\sqrt[3]{\frac{[1+13.493(\Pi_\Delta)\sqrt[3]{\text{Pr}^2}]^8}{\text{Pr}^{19/3}\,\Delta^{28}(\Pi_\Delta)^4(9\Delta-5)}} \tag{12}$$

The reliability of the present theoretical approach is verified through the few results available in literature focusing on transition in free convection. For instance, Lock and Trotter [10] found the experimental transition in water to be $(\text{Ra})_\text{T} \cong 3\cdot10^9$ while the present analysis gives a $1.825\cdot10^9$ value. More recently, from the extrapolation of a dimensional analysis by

Table 1. Laminar, transition, and turbulent parameters for an isothermal vertical wall

Pr	Δ	Π_Δ	Laminar		Turbulent		Transition threshold	
			$\frac{\delta}{x}\sqrt[4]{\mathrm{Ra}_x}$	$\frac{\mathrm{Nu}_x}{\sqrt[4]{\mathrm{Ra}_x}}$	$\frac{\delta}{x}\sqrt[10]{\mathrm{Ra}_x}$	$\frac{\mathrm{Nu}_x}{\sqrt[5]{\mathrm{Ra}_x^2}}$	$(\mathrm{Ra}_x)_\mathrm{D}$	$(\mathrm{Ra}_x)_\mathrm{T}$
0.757	1	0.0366	5.202	0.384	0.66	0.0252	$9.17\cdot10^5$	$7.721\cdot10^7$
1	0.937	0.0356	5.368	0.398	0.606	0.0244	$2.015\cdot10^6$	$1.200\cdot10^8$
2	0.811	0.0333	5.775	0.427	0.487	0.0225	$1.400\cdot10^7$	$3.367\cdot10^8$
5	0.697	0.0308	6.264	0.458	0.363	0.02	$1.715\cdot10^8$	$1.180\cdot10^8$
7	0.667	0.0301	6.42	0.467	0.325	0.0191	$4.229\cdot10^8$	$1.825\cdot10^9$
10	0.641	0.0294	6.567	0.475	0.288	0.0181	$1.092\cdot10^9$	$2.859\cdot10^9$
20	0.604	0.0285	6.777	0.489	0.227	0.0163	$6.556\cdot10^9$	$6.864\cdot10^9$
30	0.59	0.0281	6.887	0.492	0.197	0.0153	$1.891\cdot10^{10}$	$1.099\cdot10^{10}$
40	0.582	0.0278	6.928	0.496	0.178	0.0146	$3.884\cdot10^{10}$	$1.582\cdot10^{10}$
50	0.577	0.0277	6.951	0.499	0.164	0.0141	$6.766\cdot10^{10}$	$2.096\cdot10^{10}$
100	0.567	0.0274	7.065	0.499	0.128	0.0125	$4.025\cdot10^{11}$	$4.666\cdot10^{10}$
250	0.56	0.0272	7.102	0.503	0.091	0.0106	$3.885\cdot10^{12}$	$1.475\cdot10^{11}$
500	0.558	0.0271	7.115	0.504	0.071	0.0093	$2.144\cdot10^{13}$	$3.569\cdot10^{11}$

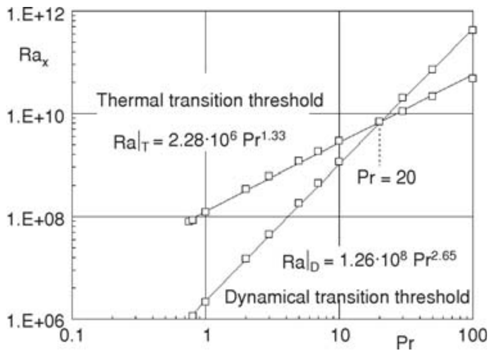

Figure 2. Mathematical transition threshold against the Prandtl number for Pr > 0.7

Arpaci and Kao [3] the transition is calculated to be $(\mathrm{Ra})_\mathrm{T} \cong 4.2\cdot10^7$ while the corresponding value in tab. 1 is $1.2\cdot10^8$. Even if the comparison is satisfactory one has to precise that the literature is much contrasted and important discrepancies exist on this subject.

Moreover, and unfortunately, from the authors knowledge, no result in literature deals with the dynamical transition from laminar to turbulence so that no comparison is possible with $(\mathrm{Ra})_\mathrm{D}$.

To access a full analysis, the corresponding theoretical transition thresholds are reported in fig. 2, and in tab. 1 as well, for a wide Prandtl number range. The equations of the best fit curves giving simple useful correlations have also been reported in fig. 2.

This theoretical analysis shows that there appears to have similarities in the mechanisms of transition from laminar to turbulent free convection flows between those occurring for isothermal walls and those producing a constant rate of heat transfer. In the present study, Pr = 20 seems to be a singular value below which the thermal transition threshold lags behind that of the dynamical one. Similar comments were made in [1] for the uniform heat flux case but the singular value was found to be Pr = 100.

Conclusions

In the way to give any advance towards an understanding of the process whereby turbulence arises and to enhance the discussion on the transition threshold knowledge the results of a theoretical investigation of the transition threshold in a free convection boundary-layer flow have been reported. As previously shown for the uniform heat flux condition, the transition for the isothermal wall condition is also Prandtl number dependent and occurs differently depending on whether the dynamical or thermal viewpoints are interested. A change in the transition threshold behaviour is observed at Pr = 20. Authors feel that this analysis and deduced correlations are a simple way to define the transition threshold and offers an engineering applicability in the field of turbulent natural convection.

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Nomenclature

a – thermal diffusivity, [m^2s^{-1}]
C_p – specific heat capacity, [$Jkg^{-1}K^{-1}$]
g – acceleration of the gravity, [ms^{-2}]
h – heat transfer coefficient, [$Wm^{-2}K^{-1}$]
k – thermal conductivity, [$Wm^{-1}K^{-1}$]
Nu – Nusselt number ($= hx/k$)
Pr – Prandtl number ($= \mu C_p/k$)
Ra – modified Rayleigh ($= q\beta\Delta Tx^3/a\vartheta$)
U – x-velocity, [ms^{-1}]
V – y-velocity, [ms^{-1}]
x, y – parallel and normal to the vertical plane, [m]

Greek symbols

β – coefficient of thermal expansion, [K^{-1}]
δ – dynamical boundary layer thickness, [m]
δ_T – thermal boundary layer thickness, [m]
Δ – thermal to velocity layer thickness ratio ($= \delta_T/\delta$)
μ – dynamic viscosity, [Pa·s]
ν – kinematic viscosity, [m^2s^{-1}]
ρ – density, [kgm^{-3}]
τ_w – shear stress, [Pa]
Θ – temperature, [°C]

Subscripts

w wall

References

[1] Varga, C., *et al*., Turbulent Boundary-Layer Buoyant Flow Modeling over a Wide Prandtl Number Range. *Acta Mechanica, 172* (2004), 1-2, pp. 65-73
[2] Popa, C. V., *et al*., On Heat Transfer in External Natural Convection Flows Using Two Nanofluids, *International Journal of Thermal Sciences, 49* (2010), 6, pp. 901-908
[3] Arpaci, V. S., Kao, S-H., Foundations of Buoyancy Driven Heat Transfer Correlations. *Journal of Heat Transfer, 123* (2001), 6, pp. 1181-1184
[4] Mladin, E. C., *et al*., Revisited Laminar Free Convection Theory by Integral Method. *Proceedings*, 9th Conferentina Nationala de Termotechnica, Craiova, Romania, 1999, Vol. II, pp. 227-238
[5] Kakac, S., Yener, Y., Convective Heat Transfer, CRC Press, Boca Raton, Fla., USA, 1995
[6] Ruckenstein, E., On the Laminar and Turbulent Free Convection Heat Transfer from a Vertical Plate over the Entire Range of Prandtl Numbers, *International Communication of Heat and Mass Transfer, 25* (1998), 7, pp. 1009-1018
[7] Polidori, G., *et al*., Extending Karman-Pohlhausen Method to Unsteady Free Convection for Pr > 0.6, *C.R. Acad. Sci. Paris, IIb 328* (2000), 10, pp. 763-766
[8] Polidori, G.,*et al*., Transient Flow Rate Behavior in an external Natural Convection Boundary Layer. *Mechanical Research Communication, 30* (2003), pp. 615-621

[9] Eckert, E. R. G., Jackson, T.W., Analysis of Turbulent Free-Convection Boundary Layer on Flat Plate, *NACA Report, 1015* (1950), pp. 1-7
[10] Lock, G. S. H., Trotter, F. J. de B., Observations on the Structure of a Turbulent Free Convection Boundary layer, *International Journal of Heat and Mass Transfer, 11* (1968), 8, pp. 1225-1232

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