Open forum

A SHORT REMARK ON STEWART 1962 VARIATIONAL PRINCIPLE FOR LAMINAR FLOW IN A UNIFORM DUCT

by

Hong-Yan LIU [a,c*], ***Lei ZHAO*** [b], ***and Hong-Yan TU*** [a]

[a] School of Fashion Technology, Zhongyuan University of Technology, Zhengzhou, China
[b] School of Textile and Garment, Yancheng Institute of Industry Technology, Yancheng, China
[c] National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University, Suzhou, China

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This paper concludes that Stewart 1962 variational principle for laminar flow in a uniform duct is for a differential-difference. Some generalized variational principles are elucidated with or without Stewart's discrete treatment.

Key words: *variational principle, differential-difference equation, semi-inverse method, viscous flow*

Introduction

Recently the inverse problem of calculus of variations became a hot topic as shown in the Open Forum in this journal [1-3]. The inverse problem is to search for a variational principle from the governing equations and boundary conditions, and the Euler-Lagrange equations are the governing equations. There has been a great deal of activity in the study of this problem since the early 20th century. A notable advance in this field was the semi-inverse method proposed by the famous Chinese mathematician, Dr. Ji-Huan He [4]. In this paper we will re-studied an old variational problem in view of Dr. He's approach.

Stewart 1962 variational principle

In 1962 Stewart studied a steady Newtonian flow of constant viscosity and density in a long, cylindrical duct of arbitrary cross section S under a known pressure gradient [5]. The governing equations for this problem are [5]:

$$\frac{\partial u_z}{\partial z} = 0 \quad (1)$$

$$\frac{\partial p}{\partial x} = 0 \quad (2)$$

$$\frac{\partial p}{\partial y} = 0 \quad (3)$$

* Corresponding author; e-mail: phdliuhongyan@gmail.com

$$-\frac{\partial p}{\partial z}+\frac{\partial t_{xz}}{\partial x}+\frac{\partial t_{yz}}{\partial y}=0 \tag{4}$$

The viscous stress tensor are given by:

$$t_{xz}=\mu\frac{\partial u_z}{\partial x} \tag{5}$$

$$t_{yz}=\mu\frac{\partial u_z}{\partial y} \tag{6}$$

The boundary conditions are:

$$p(0)=p_0,\quad p(L)=p_L \tag{7}$$

where u_z is the local fluid velocity in the z direction, and p – the pressure.

Combining eqs. (4), (5), and (6) together, we have:

$$-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}\right)=0 \tag{8}$$

Stewart obtained the variational principle [5]

$$J(u_z)=\frac{\mu L}{2}\iint_S\left[\left(\frac{\partial u_z}{\partial x}\right)^2+\left(\frac{\partial u_z}{\partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y+(p_\mathrm{L}-p_0)\iint_S u_z\mathrm{d}x\mathrm{d}y \tag{9}$$

The semi-inverse method

Stewart 1962 variational principle can be obtained from the following differential-difference equation:

$$-\frac{p_\mathrm{L}-p_0}{L}+\mu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}\right)=0 \tag{10}$$

This variational principle is, of course, correct, but it is subject to the constraint of eq. (1). The variational principle for the differential-difference system was completely studied in [6].

By the semi-inverse method [1-3,7-9], we can assume the trial-functional:

$$J(u_z,p)=\iint_S\left(\frac{\partial p}{\partial z}u_z+F\right)\mathrm{d}x\mathrm{d}y \tag{11}$$

where F is unknown function to be further determined, which is free from p and its derivatives. The advantage of the trial-functional is that the stationary condition with respect to p results in eq. (1). After identification of F in eq. (11), a two-function (u_z, p) variational principle can be obtained which reads:

$$J(u_z,p)=\iint_S\left\{\frac{\partial p}{\partial z}u_z+\frac{\mu}{2}\left[\left(\frac{\partial u_z}{\partial x}\right)^2+\left(\frac{\partial u_z}{\partial y}\right)^2\right]\right\}\mathrm{d}x\mathrm{d}y \tag{12}$$

It is obvious that the stationary conditions of eq. (12) are eq. (1) and eq. (8).

Similarly by the semi-inverse method [1-3, 7-9] and Stewart's discrete treatment for $\partial p/\partial z$, a three-function (t_{xz}, t_{yz}, p) variational principle can be obtained which reads:

$$J(t_{xz},t_{yz},p)=\iint_S\left[p\left(\frac{\partial t_{xz}}{\partial x}+\frac{\partial t_{yz}}{\partial y}\right)-\frac{p(p_\mathrm{L}-p_0)}{L}\right]\mathrm{d}x\mathrm{d}y \tag{13}$$

A four-function (u_z, p, t_{xz}, t_{yz}) variational principle can be also obtained which reads:

$$J(u_z, p, t_{xz}, t_{yz}) = a\iint_S \left[\left(-\frac{\partial p}{\partial z} + \frac{\partial t_{xz}}{\partial x} + \frac{\partial t_{yz}}{\partial y}\right)u_z\right]\mathrm{d}x\mathrm{d}y \tag{14a}$$

or

$$J(u_z, p, t_{xz}, t_{yz}) = a\iint_S \left(p\frac{\partial u_z}{\partial z} - \tau_{xz}\frac{\partial u_z}{\partial x} - \tau_{yz}\frac{\partial u_z}{\partial y}\right)\mathrm{d}x\mathrm{d}y \tag{14b}$$

Conclusion

We re-study the Stewart 1962 variational principle using the semi-inverse method, and some extensions are obtained. The semi-inverse method is a powerful tool to the search for variational formulations directly from governing equations.

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