A SHORT REMARK ON STEWART 1962 VARIATIONAL PRINCIPLE FOR LAMINAR FLOW IN A UNIFORM DUCT

by

Hong-Yan LIU a,c*, Lei ZHAO b, and Hong-Yan TU a

a School of Fashion Technology, Zhongyuan University of Technology, Zhengzhou, China
b School of Textile and Garment, Yancheng Institute of Industry Technology, Yancheng, China
c National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University, Suzhou, China

Original scientific paper
DOI: 10.2298/TSCI140321063L

This paper concludes that Stewart 1962 variational principle for laminar flow in a uniform duct is for a differential-difference. Some generalized variational principles are elucidated with or without Stewart’s discrete treatment.

Key words: variational principle, differential-difference equation, semi-inverse method, viscous flow

Introduction

Recently the inverse problem of calculus of variations became a hot topic as shown in the Open Forum in this journal [1-3]. The inverse problem is to search for a variational principle from the governing equations and boundary conditions, and the Euler-Lagrange equations are the governing equations. There has been a great deal of activity in the study of this problem since the early 20th century. A notable advance in this field was the semi-inverse method proposed by the famous Chinese mathematician, Dr. Ji-Huan He [4]. In this paper we will re-studied an old variational problem in view of Dr. He's approach.

Stewart 1962 variational principle

In 1962 Stewart studied a steady Newtonian flow of constant viscosity and density in a long, cylindrical duct of arbitrary cross section S under a known pressure gradient [5]. The governing equations for this problem are [5]:

\[ \frac{\partial u_z}{\partial z} = 0 \]  
\[ \frac{\partial p}{\partial x} = 0 \]  
\[ \frac{\partial p}{\partial y} = 0 \]

* Corresponding author; e-mail: phdluihongyan@gmail.com
The viscous stress tensor are given by:
\[
\begin{align*}
\sigma_{xz} &= \mu \frac{\partial u_z}{\partial x} \\
\sigma_{yz} &= \mu \frac{\partial u_z}{\partial y}
\end{align*}
\] (5)

The boundary conditions are:
\[
p(0) = p_0, \quad p(L) = p_L
\] (7)

where \(u_z\) is the local fluid velocity in the \(z\) direction, and \(p\) – the pressure.

Combining eqs. (4), (5), and (6) together, we have:
\[
-\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} \right) = 0
\] (8)

Stewart obtained the variational principle [5]
\[
J(u_z) = \frac{\mu L}{2} \int_S \left[ \left( \frac{\partial u_z}{\partial x} \right)^2 + \left( \frac{\partial u_z}{\partial y} \right)^2 \right] \, dx dy + (p_L - p_0) \int_S u_z \, dx dy
\] (9)

The semi-inverse method

Stewart 1962 variational principle can be obtained from the following differential-difference equation:
\[
-\frac{p_L - p_0}{L} + \mu \left( \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} \right) = 0
\] (10)

This variational principle is, of course, correct, but it is subject to the constraint of eq. (1). The variational principle for the differential-difference system was completely studied in [6].

By the semi-inverse method [1-3, 7-9], we can assume the trial-functional:
\[
J(u_z, p) = \int_S \left( \frac{\partial p}{\partial z} u_z + F \right) \, dx dy
\] (11)

where \(F\) is unknown function to be further determined, which is free from \(p\) and its derivatives.

The advantage of the trial-functional is that the stationary condition with respect to \(p\) results in eq. (1). After identification of \(F\) in eq. (11), a two-function \((u_z, p)\) variational principle can be obtained which reads:
\[
J(u_z, p) = \int_S \left[ \frac{\partial p}{\partial z} u_z + \mu \left( \frac{\partial u_z}{\partial x} \right)^2 + \left( \frac{\partial u_z}{\partial y} \right)^2 \right] \, dx dy
\] (12)

It is obvious that the stationary conditions of eq. (12) are eq. (1) and eq. (8).

Similarly by the semi-inverse method [1-3, 7-9] and Stewart’s discrete treatment for \(\partial p/\partial z\), a three-function \((t_{xz}, t_{yz}, p)\) variational principle can be obtained which reads:
\[
J(t_{xz}, t_{yz}, p) = \int_S \left[ \mu \left( \frac{\partial t_{xz}}{\partial x} + \frac{\partial t_{yz}}{\partial y} \right) - p(p_L - p_0) \right] \, dx dy
\] (13)
A four-function \((u, p, t_x, t_y)\) variational principle can be also obtained which reads:

\[
J(u, p, t_x, t_y) = \frac{1}{2} \int \left[ -\frac{\partial p}{\partial z} + \frac{\partial t_x}{\partial x} + \frac{\partial t_y}{\partial y} \right] u \, dx \, dy
\] (14a)

or

\[
J(u, p, t_x, t_y) = \frac{1}{2} \int \left[ p \frac{\partial u_x}{\partial z} + \tau_x - \tau_z \frac{\partial u_x}{\partial x} - \tau_z \frac{\partial u_x}{\partial y} \right] dx \, dy
\] (14b)

**Conclusion**

We re-study the Stewart 1962 variational principle using the semi-inverse method, and some extensions are obtained. The semi-inverse method is a powerful tool to the search for variational formulations directly from governing equations.

**Acknowledgments**

The work is supported by Jiangsu Planned Projects for Postdoctoral Research Funds under grant No. 1401076B, National Natural Science Foundation of China under grant No.11372205 and Project for Six Kinds of Top Talents in Jiangsu Province under grant No. ZBZZ-035, Science & Technology Pillar Program of Jiangsu Province under grant No. BE2013072 China Postdoctoral Science Foundation under grant No. 2015M571806, China National Textile And Apparel Council Project under grant No. 2015011, Key Scientific Research Projects of Henan Province under grant No. 16A540001.

**References**


