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VARIATIONAL PRINCIPLE FOR UNSTEADY HEAT CONDUCTION EQUATION

by

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Short paper
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The semi-inverse method is used to establish a variational principle for an unsteady heat conduction equation.

Key words: *variational principle, heat conduction, semi-inverse method*

Introduction

Recently variational principles for thermal problems became a hot topic [1-5], and the He-Lee's variational principle for heat conduction has been completely discussed [1-4]. In this paper we will study an unsteady heat conduction equation of hyperbolic type [6]:

$$c\rho \frac{\partial T}{\partial t} - \frac{\partial q}{\partial x} = 0 \tag{1}$$

$$t_0 \frac{\partial q}{\partial t} + q + k \frac{\partial T}{\partial x} = 0 \tag{2}$$

where T is the temperature, q – the specific rate of heat flow, t_0 – the relaxation time, c – the specific heat coefficient, ρ – the density, and k – the heat conduction coefficient.

Obviously, with $t_0 = 0$, the studied problem reduces to the classical Fourier's heat conduction.

Liu established a variational principle for 3-D unsteady heat conduction [6]. This paper tends to continue open forum paper discussion in issue 5, 2013, of the journal *Thermal Science* using the semi-inverse method [7, 8].

He's semi-inverse method

The semi-inverse method was first proposed by the famous Chinese mathematician, Dr. Ji-Huan He, in 2007 [7], a historical remark on the method was illustrated in [3].

Combining eqs. (1) and (2) together, we obtain:

$$c\rho t_0 \frac{\partial^2 q}{\partial t^2} + c\rho \frac{\partial q}{\partial t} + k \frac{\partial^2 q}{\partial x^2} = 0 \tag{3}$$

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To begin with establishment of the needed variational principle for eq. (3) or eqs. (1) and (2), we first consider a functional:

$$J(q)=\iint\left[\frac{1}{2}c\rho t_0\left(\frac{\partial q}{\partial t}\right)^2+\frac{1}{2}k\left(\frac{\partial q}{\partial x}\right)^2\right]\mathrm{d}t\mathrm{d}x \tag{4}$$

Stationary condition can be easily obtained:

$$c\rho t_0\frac{\partial^2 q}{\partial t^2}+k\frac{\partial^2 q}{\partial x^2}=0 \tag{5}$$

Comparing eq. (5) with eq. (3), we can find that the term, $c\rho\partial\theta/\partial t$, is missed. Using the semi-inverse method [7-11], we can construct a trial functional in the form:

$$J(q)=\iint\left\{\left[\frac{1}{2}c\rho t_0\left(\frac{\partial q}{\partial t}\right)^2+\frac{1}{2}k\left(\frac{\partial q}{\partial x}\right)^2\right]f(t)+F(q)\right\}\mathrm{d}t\mathrm{d}x \tag{6}$$

where f is an unknown function of t, and F is an known function of q and/or its derivatives.

Making eq. (6) stationary with respect to q, we can obtain the following Euler-Lagrange equation:

$$-\frac{\partial q}{\partial t}\left(c\rho t_0 f\frac{\partial q}{\partial t}\right)-\frac{\partial}{\partial x}\left(kf\frac{\partial q}{\partial t}\right)+\frac{\delta F}{\delta q}=0 \tag{7}$$

or

$$-c\rho t_0 f\frac{\partial^2 q}{\partial t^2}-c\rho t_0\frac{\partial q}{\partial t}\frac{\partial f}{\partial t}-kf\frac{\partial^2 q}{\partial x^2}+\frac{\delta F}{\delta q}=0 \tag{8}$$

where $\delta F/\delta q$ is He's derivative defined as [9]:

$$\frac{\delta F}{\delta q}=\frac{\partial F}{\partial q}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial q_x}\right)-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial q_{\mathrm{t}}}\right) \tag{9}$$

Comparing eq. (8) with eq. (3), we set:

$$\frac{\partial f}{\partial t}=\frac{1}{t_0 f},\quad \frac{\partial F}{\partial q}=0 \tag{10}$$

From eq. (10), we can identify f and F:

$$f=\exp\left(\frac{t}{t_0}\right),\quad F=0 \tag{11}$$

We, therefore, obtain the following variational principle:

$$J(q)=\iint\left\{\left[\frac{1}{2}c\rho t_0\left(\frac{\partial q}{\partial t}\right)^2+\frac{1}{2}k\left(\frac{\partial q}{\partial x}\right)^2\right]\exp\left(\frac{t}{t_0}\right)\right\}\mathrm{d}t\mathrm{d}x \tag{12}$$

Conclusion

The paper shows that the semi-inverse method is a powerful tool to establishment of a needed variational principle directly from the governing equations.

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