MAGNETOHYDRODYNAMIC THIN FILM AND HEAT TRANSFER OF POWER LAW FLUIDS OVER AN UNSTEADY STRETCHING SHEET WITH VARIABLE THERMAL CONDUCTIVITY

by

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This paper presents an investigation on the magnetohydrodynamic thin film flow and heat transfer of a power law fluid over an unsteady stretching sheet. The effects of power law viscosity on a temperature field are taken into account with a modified Fourier's law proposed by assuming that the temperature field is similar to the velocity field. The governing equations are reduced to a system of non-linear ordinary differential equations. The numerical solutions are obtained by using the shooting method coupled with the Runge-Kutta method. The influence of the Hartmann number, the power law exponent, the unsteadiness parameter, the thickness parameter and the generalized Prandtl number on the velocity and temperature fields are presented graphically and analyzed. Moreover, the critical formula for parameters, are derived which indicated that the magnetic field has no effect on the critical value.

Key words: *power law fluids, thin film, stretching sheet, magnetohydrodynamics, heat transfer*

Introduction

In recent years, the analysis of fluid flow and heat transfer on a thin liquid film has attracted considerable attention in many fields of science and technology due to its wide applications, such as wire and fiber coating, metal and polymer extrusion, foodstuff processing, continuous casting, drawing of plastic sheets, exchangers, and chemical processing equipment. The flow and heat transfer in a finite liquid film over a continuous surface is important in physical meaning. Wang [1] firstly studied the flow problem within a finite film of Newtonian fluid over an unsteady stretching sheet. Following his pioneering work, Andersson *et al.* [2] studied the flow of a thin liquid film of an incompressible fluid obeying the power law model. Furthermore, Andersson *et al.* [3], Liu and Andersson [4] analyzed the momentum and heat transfer in a laminar liquid film on a horizontal stretching sheet. In their work, the multiple shooting subroutine MUSN was used to solve the non-linear differential equations. Chen [5, 6] investigated the problem of momentum and heat transfer in a thin liquid film of power law fluid on an unsteady stretching surface while the effect of viscous dissipation was taken into account. Later, Chen [7] examined the effect of Marangoni convection on the flow and

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heat transfer within a power law liquid film on an unsteady stretching sheet. By means of the homotopy analysis method (HAM), Wang [8], and Wang and Pop [9] considered the flow of a power law fluid film on an unsteady stretching surface, the critical value of the unsteadiness parameter was derived in [9]. Huang *et al.* [10] investigated the effect of thermoviscosity on the heat transfer in a power law liquid film over an unsteady stretching sheet, the film thickness, the temperature distributions, the local heat transfer rate, the local skin-friction coefficient were obtained using the Chebyshev finite difference method. Vajravelu *et al.* [11] studied the effects of viscous dissipation and the temperature-dependent thermal conductivity on an unsteady flow and heat transfer in a thin liquid film of a non-Newtonian Ostwald-de Waele fluid over a horizontal porous stretching surface. The governing equations of the problem were solved by the Keller-Box method. Dandapat *et al.* [12] analyzed the influence of thermocapillarity on the flow and heat transfer in a thin liquid film on a horizontal stretching sheet. Later, Noor and Hashim [13] examined the effects of thermocapillarity and a magnetic field by HAM. Dandapat *et al.* [14], Abel *et al.* [15, 16], Nandeppannavar *et al.* [17], Aziz *et al.* [18], and Hussan *et al.* [19] investigated the effects of variable viscosity, variable thermal conductivity, non-uniform heat source, viscous dissipation, external magnetic field, thermal radiation, and concentration on the flow and heat transfer of a thin flow over an unsteady stretching. Abbas *et al.* [20] studied the flow problem in a thin liquid film of second grade fluid over an unsteady stretching surface. Rashidi and Keimanesh [21] presented magnetohydrodynamic (MHD) flow in a laminar liquid film above a horizontal stretching surface by using the differential transform method and Pade approximant. Aziz and Hashim [22] and Aziz *et al.* [23] analyzed the effects of viscous dissipation and internal heat generation on flow and heat transfer in a thin film on an unsteady stretching sheet and it should be noted that a general surface temperature was taken into consideration. Furthermore, the problem of flow and heat transfer within a finite thin film over an unsteady stretching sheet are extended by Bachok *et al.* [24], and Xu *et al.* [25] with consideration of nanoliquid film. Abdel-Rahman [26] studied the unsteady flow and heat transfer phenomena in a power law fluid over a porous stretching surface, taking into account the variable thermal conductivities and variable viscosities. Khalal *et al.* [27] analyzed the mixed convection heat and mass transfer processes in moving turbulent binary liquid film.

Motivated by the previous mentioned works, we consider in this paper the MHD thin film flow and heat transfer of a power law fluid over an unsteady stretching sheet. The effects of power law viscosity on a temperature field are taken into account with a modified Fourier's law proposed by Zheng by assuming that the temperature field is similar to the velocity field [28-33]. External magnetic field on the flow and heat transfer of non-Newtonian fluids within a finite thin liquid film on an unsteady stretching sheet is also considered in the governing equations. The governing equations are reduced to a system of non-linear ordinary differential equations. The numerical solutions are obtained by using BVP4C in MATLAB, the shooting method coupled with the Runge-Kutta method and the Newtonian technique. The influence of Hartmann number, H, power law index, unsteadiness parameter, thickness parameter, the generalized Prandtl number, Pr, on the velocity and temperature fields are presented graphically and discussed.

Formulation of the problem

Consider the flow and heat transfer in a thin, power law liquid film over a horizontal sheet which issues from a narrow slot and the fluid motion is caused by stretching the elastic sheet. A schematic of the physical model and co-ordinate system is shown in fig. 1. A magnetic field $B = B_0(1-at)^{-1/2}$ normal to the stretching sheet is applied. The time-dependent governing equations for mass, momentum, and energy conservation are given by:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

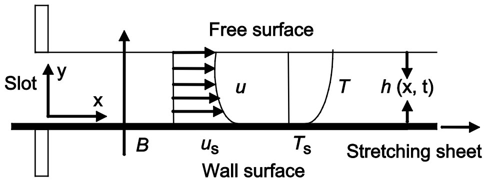

Figure 1. Schematic of the physical system

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\rho}\frac{\partial \tau}{\partial y}-\frac{1}{\rho}\delta B^2 u \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{\rho C_p}\frac{\partial}{\partial y}\left[k(T)\frac{\partial T}{\partial y}\right] \tag{3}$$

where u and v are the velocity components along the x- and y-directions, respectively, τ is the shear stress, ρ – the density, and C_p – the specific heat at constant pressure. In the present study, we assume the shear stress:

$$\tau = K\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y} \tag{4}$$

where K is the consistency viscosity coefficient, t - the time, and n - the power law index. The case $n=1$ corresponds to a Newtonian fluid, $0<n<1$ is descriptive of a pseudo-plastic fluid while $n>1$ describes a dilatant fluid. In the present study, the thermal conductivity of non-Newtonian fluids is assumed to vary power-law-dependent with the temperature gradient [28-33]:

$$k(T) = k_0\left|\frac{\partial T}{\partial y}\right|^{n-1} \tag{5}$$

where k_0 is the consistency thermal coefficient, and T – the temperature.

Assuming that the interface of the liquid film is smooth and free of surface waves and that the viscous shear stress and the heat flux vanish at the adiabatic free surface, the boundary conditions become:

$$y=0: u=u_s, \qquad v=0, T=T_s \tag{6}$$

$$y\to h(x,t): \frac{\partial u}{\partial y}=\frac{\partial T}{\partial y}=0, \qquad v=u\frac{\partial h}{\partial x}+\frac{\partial h}{\partial t} \tag{7}$$

where the sheet continuously moves in its own plane with the velocity u_s and the surface temperature of the stretching sheet varies with the horizontal co-ordinate x and time *t:*

$$u_s = bx(1-at)^{-1}, \qquad T_s = T_0 - T_{\text{ref}}dbx(1-at)^{-1} \tag{8}$$

where a and b are positive constants, T_0 is the temperature at the origin, d – the positive constant of proportionality with dimension [time/length], and T_{ref} – the reference temperature which can be taken as a constant reference temperature. It should be noted that the expressions given by the eqs. (8) and (9) are valid for time $t < a^{-1}$.

In terms of the standard definition of the stream function such that $u=\partial\psi/\partial y$ and $v=-\partial\psi/\partial x$ and introducing the following transformation variables:

$$\psi = (b^{1-2n}\gamma^{-1})^{-\frac{1}{n+1}} x^{\frac{2n}{n+1}} (1-at)^{\frac{1-2n}{n+1}} f(\eta), \qquad T = T_0 - T_{\text{ref}} \frac{dbx}{1-at}\theta(\eta) \tag{9}$$

$$\eta = (b^{2-n}\gamma^{-1})^{\frac{1}{n+1}} x^{\frac{1-n}{n+1}} (1-at)^{\frac{n-2}{n+1}} y, \qquad \beta = (b^{2-n}\gamma^{-1})^{\frac{1}{n+1}} (1-at)^{\frac{n-2}{n+1}} h(x,t) \tag{10}$$

$$\text{H} = \frac{\delta}{b\rho} B_0^{\,2}, \qquad S = \frac{a}{b}, \qquad \gamma = \frac{K}{\rho} \tag{11}$$

$$\text{Re}_x = u_s^{\,2-n} x^n \gamma^{-1} = b^{2-n} x^2 \gamma^{-1} (1-at)^{n-2}, \qquad \text{Pr} = KC_p k_0^{\,-1} T_{\text{ref}}^{\,1-n} d^{1-n} \tag{12}$$

The governing eqs. (1)-(3), (6), and (7) are converted into:

$$(|f''|^{n-1} f'')' - S\left(\frac{2-n}{n+1}\eta f'' + f'\right) + \left(\frac{2n}{n+1} ff'' - f'^2 - \text{H}f'\right) = 0 \tag{13}$$

$$(|\theta'|^{n-1}\theta')' + \text{Pr}\left[\left(\frac{2n}{n+1} f\theta' - f'\theta\right) - S\left(\frac{2-n}{n+1}\eta\theta' + \theta\right)\right] = 0 \tag{14}$$

$$f(0)=0, \quad f'(0)=1, \quad f''(\beta)=0, \quad f(\beta)=\frac{2-n}{2n}\beta S, \quad \theta(0)=1, \quad \theta'(\beta)=0 \tag{15}$$

where S is the unsteadiness parameter, Pr – the generalized Prandtl number, β – the thickness of the liquid film, Re_x – the local Reynolds number, and H – the Hartmann number.

Numerical methods

In order to obtain numerical solutions, we transfer eqs. (13)-(15) to a system of first order differential equations by denoting the f', f'', and θ' using variables p, q, and r, respectively:

$$\frac{\text{d}f}{\text{d}\eta} = p, \qquad \frac{\text{d}p}{\text{d}\eta} = q, \qquad \frac{\text{d}q}{\text{d}\eta} = \frac{1}{n}|q|^{1-n}\left[S\left(\frac{2-n}{n+1}\eta q + p\right) + p^2 - \frac{2n}{n+1} fq + \text{H}p\right] \tag{16}$$

$$\frac{\text{d}\theta}{\text{d}\eta} = r, \qquad \frac{\text{d}r}{\text{d}\eta} = \frac{1}{n}\text{Pr}|r|^{1-n}\left[\left(p\theta - \frac{2n}{n+1} fr\right) + S\left(\theta + \frac{2-n}{n+1}\eta r\right)\right] \tag{17}$$

$$f(0)=0, \quad p(0)=1, \quad q(\beta)=0, \quad f(\beta)=\frac{2-n}{2n}\beta S, \quad \theta(0)=1, \quad r(\beta)=0 \tag{18}$$

Equations (16) and (17) have five first order ordinary differential equations and they have six boundary value conditions (18). So there exists correspondence between the parameters β and S. The previous boundary value problem (16) and (18): $f(0) = 0$, $p(0) = 1$, $q(\beta) = 0$, $f(\beta) = (2 - \text{n})\,\beta S/2n$ is solved by the shooting method coupled with the Newtonian scheme for a known value of S. The initial guess value of β is given by the program BVP4C in MATLAB. And the value of β is adjusted so that the condition (18): $f(\beta) = (2 - \text{n})\,\beta S/2n$ holds: This is done on a trial and error basis. Then, the ordinary differential eqs. (16)-(18) are solved by using the standard fourth Runge-Kutta method and shooting method for a known value of S and β.

Results and analysis

The non-linear boundary value problem (13)-(15) is solved by the shooting method coupled with Newton method and BVP4C in the section *Numerical methods*. To verify the accuracy and effectiveness of the present method, the results for degradation (H = 0) are presented with those in references [9, 5-7, 15, 16] in tab. 1. It is seen that the present results are agree very well with the references.

Table 1. Comparison of β and $-f''(0)$ for various values of n and S with H = 0

n	S	Andersson *et al.* [3]		Chen [5-7]		Wang and Pop [9]		Huang *et al.* [10]		Present results	
		β	$-f''(0)$	β	$-f''(0)$	β	$-f''(0)$	β	$-f''(0)$	β	$-f''(0)$
0.8	0.8			1.3267	1.2229	1.32520	1.22245	1.3268	1.22316	1.3272	1.2230
0.8	1.0					0.82544	1.09639			0.8168	1.1111
0.8	1.2			0.4297	0.7734	0.43011	0.77887	0.4325	0.77927	0.4326	0.7791
1.0	0.8	2.15199	1.24581			2.15199	1.24580	2.15199	1.24581	2.1522	1.2457
1.0	1.0					1.54362	1.27777			1.5438	1.2777
1.0	1.2	1.12778	1.27917			1.12778	1.27918	1.12778	1.27917	1.1280	1.2790
1.2	0.8			3.0306	1.2178	3.03484	1.21588	3.0307	1.21778	3.0310	1.2177
1.2	1.0									2.3190	1.2664
1.2	1.2			1.8276	1.3088	1.82799	1.30745	1.8278	1.30882	1.8280	1.3087

Table 2. Comparison of the critical value for various values of n

n	Wang and Pop [9]	Wang [1]	Andersson *et al.* [2]	Chen [5-7]	Huang *et al.* [10]	Present results
0.8	4/3		1.67	1.35	1.34	1.333
1.0	2	2	2.00	2.00	2	2.000
1.2	3		2.50	3.03	3	3.000

For the hydrodynamic problem, there exists a critical value of the unsteadiness parameter S_0 above which no solution could be obtained [5-7, 15]. For positive values of S, $S \to 0$ stands for the case of $\beta \to +\infty$, while $S \to S_0$ stands for the case of $\beta \to 0$. Wang and Pop [9] proposed that the critical value of the unsteadiness parameter of a power law liquid film can be given by $S_0 = 2n/(2-n)$. We also can get the critical value $S_0 = 2n/(2-n)$ (see Appendix) which indicates that the magnetic field has no effect on critical value. In other word, the critical value is the same for different values of the Hartmann number. Then, the values of the critical value obtained are compared with previous results [7, 8, 5-7, 15, 16] for the case H = 0 in tab. 2. It should be noted that the results compare favorably except for the computations of Andersson *et al.* [2] due to the incorrect kinematic conditions applied, which was mentioned in Chen [5-7]. The effects of the Hartmann number, the power law index, the unsteadiness parameter, the thickness parameter, and the generalized Prandtl number on the velocity and temperature fields are analyzed and discussed in detail further in the paper.

Figure 2 illustrates the film thickness, β, varying with the unsteadiness parameter, S, at selected values of the power law index, n, for H = 2.0. As the unsteadiness parameter increases from $0 \to S_0$, the film thickness decreases from $+\infty \to 0$. In general, the film thickness increases as the power law index increases. Figure 3 illustrates the film thickness, β, varying with the unsteadiness, S, at selected values of the Hartmann number when $n = 0.8$. As the Hartmann number increases, the film thickness decreases. It should be noted that the critical value is the same for different values of the Hartmann number. This result has been obtained from analytic results of the critical value of the unsteadiness parameter.

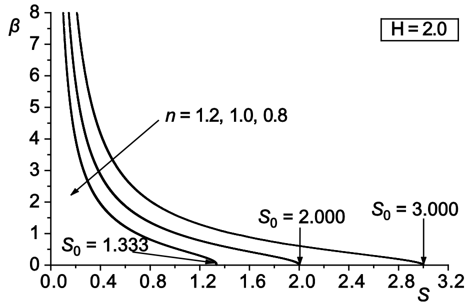

Figure 2. Variations of film thickness with the unsteadiness parameter at selected values of *n*

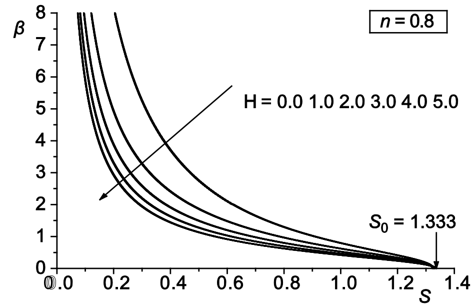

Figure 3. Variations of film thickness with the unsteadiness parameter at selected values of Hartmann number

Figures 4 and 5 show the influence of the Hartmann number on the velocity and the temperature fields for $n = 0.8$, $S = 1.0$, and Pr = 1.0. Application of a transverse magnetic field results in a drag-like force called the Lorentz force, which tends to slow down the movement of the fluid along surface and to increase the temperature. This is evident in the decreases in the velocity and increases in the temperature as the Hartmann number increases. Figures 4 and 5 show that the velocity decreases, but that the temperature increases as the Hartmann number increases. In addition, the results show that the film thickness decreases with increasing values of the Hartmann number.

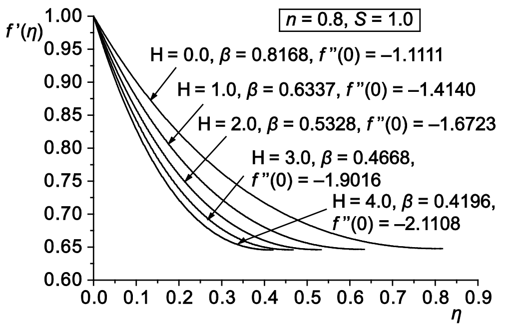

Figure 4. Effects of the Hartmann number on the velocity profiles

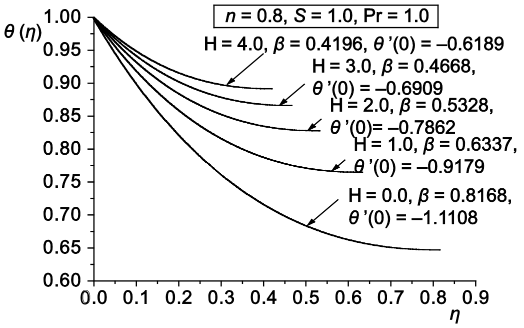

Figure 5. Effects of the Hartmann number on the temperature profiles

Figure 6 illustrates the wall shear stress parameter, $f''(0)$, with the film thickness, β, for H = 2.0 and different values of the power law index, n. It can be seen that $-f''(0)$ first increases with β rapidly, reaching a maximum, and then decreases gradually as β approaches $+\infty$. In addition, the maximum of $-f''(0)$ decreases as the power law index increases. Figure 7 demonstrates the influence of the power law index on the temperature. The modified Fourier's law for power-law thin film, proposed by Zheng *et al.* [30, 33] and Li *et al.* [31, 32] is taken into account in this study. It can be seen that $\theta(\eta)$ decreases from $1 \to \theta(\beta)$ as η increases from $0 \to \beta$. So the dimensionless temperature decreases as y from the wall surface to the free surface. Further, the film thickness decreases and the temperature vary margin $(|\theta(\beta)-\theta(0)|)$ increases as the power law index increases. Figure 8 displays the wall surface heat fluxes $-\theta(0)$ with the modified Prandtl number for S = 1.0, H = 2.0, and different values of the power law index. It shows that $-\theta(0)$ increases as Prandtl increases for all selected values of n. In addition, $-\theta(0)$ increases for $\Pr \to 0$ while $-\theta(0)$ decreases for $\Pr \to +\infty$ as the power law index increases.

Figure 9 illustrates the influence of the Prandtl number on the temperature when S = 1.0, H = 2.0, and the shear thinning sheet n = 0.8. The results show that the temperature decreases with the distance from the wall surface for the Prandtl number. The temperature also decreases as the Prandtl number increases. A nearly uniform distribution, $\theta(\eta) = 1$ or $T = T_0$, is observed in the liquid film at a very low generalized Prandtl number $\Pr \to 0$ (*e. g.* $\Pr = 0.01$).

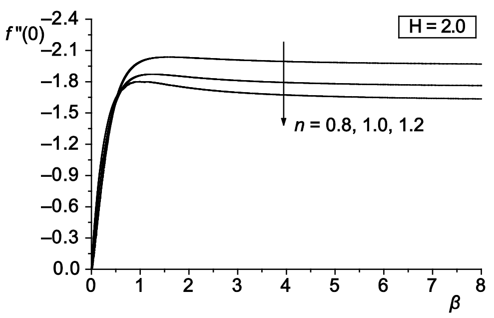

Figure 6. Variations of *f″*(0) with the film thickness at selected values of *n*

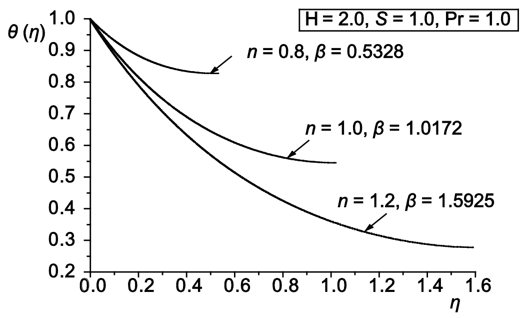

Figure 7. Effects of the power law index on the temperature profiles

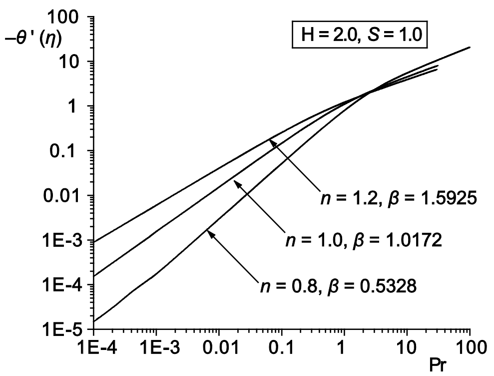

Figure 8. Variations of –θ′(0) with the Pr at selected values of *n*

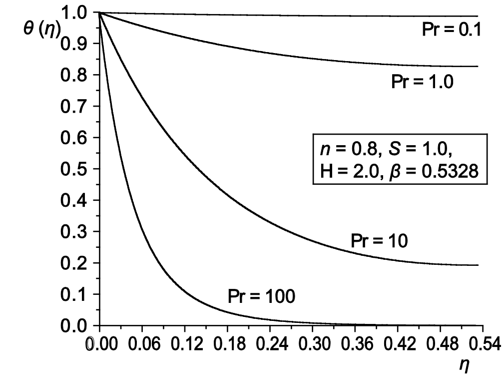

Figure 9. Effects of the Pr on the temperature profiles

Conclusions

Magnetic field effects in the power law finite thin film over an unsteady stretching sheet with variable thermal conductivity were studied in this paper. The governing partial differential equations were transformed into a system of strong non-linear ordinary differential equations using a similarity transformation. These non-linear ordinary differential equations with the boundary conditions were solved numerically using the program BVP4C in

MATLAB and the shooting method coupled with the Runge-Kutta technique. The following results are established.

- The magnetic field tends to slow down the velocity and to increase the temperature of the power-law fluids. As the strength of the magnetic field increases (the Hartmann number increases), the thickness and velocity of the thin film decrease while the temperature increases.
- The effects of power law viscosity on a temperature field are taken into account by assuming that the temperature field is similar to the velocity field with modified Fourier's law for power-law fluids (proposed by Zheng). The results indicated that the thickness of the liquid thin film decrease with decreasing in power-law index.
- The temperature decreases as the generalized Prandtl number increases and the shearing thinning film shows a smaller thickness than the shear thickening film.
- It should be noted that the critical value is $S_0 = 2n/(2 - n)$ and the critical value is the same for different values of the Hartmann number.

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Nomenclature

a, b – positive constants, [s^{-1}]
B – uniform external magnetic field, [$-kg^{0.5}S^{-0.5}s^{-0.5}m^{-1.5}$]
B_0 – positive constant, [$-kg^{0.5}S^{-0.5}s^{-0.5}m^{-1.5}$]
f – dimensionless stream function, [–]
h – film thickness, [m]
H – the Hartmann number, [–]
K – positive constant, [$kgm^{-1} s^{n-2}$]
$k(T)$ – the consistency thermal coefficient, [$Ns^{-1}K^{-1}$]
k_0 – positive constant, [$Jm^nK^{-n}s^{-1}$]
n – power-law index, [–]
Pr – generalized Prandtl number, [–]
Re_x – local Reynolds number, [–]
S – unsteadiness parameter, [–]
S_0 – critical value, [–]
T – temperature, [K]
T_0 – temperature at the origin, [K]
T_{ref} – reference temperature, [K]
T_s – temperature at the wall surface, [K]
t – time, [s]
u, v – fluid velocity component in x- and y-directions, [ms^{-1}]
u_s – velocity of the sheet, [ms^{-1}]
x, y – streamwise co-ordinate and cross-stream co-ordinate, [m]

Greek symbols

β – thickness of the liquid film, [–]
δ – electrical conductivity, [S]
η – similarity variable, [–]
θ – dimensionless temperature, [–]
ρ – density, [kgm^{-3}]
ρC_p – heat capacity, [$JK^{-1}m^{-3}$]
τ – shear stress, [Nm^{-2}]
ψ – stream function, [–]

Subscripts

s – surface

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Appendix

In order to analyze the critical value of the model with a magnetic field effect, we introduce the following transformation variable and function:

$$\eta = \beta\xi, \qquad f(\eta) = g(\xi) \tag{19}$$

$$\left\{ \frac{\mathrm{d}f}{\mathrm{d}\eta} = \frac{\mathrm{d}f}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}\eta} = \frac{1}{\beta}\frac{\mathrm{d}g}{\mathrm{d}\xi}, \quad \frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} = \frac{\mathrm{d}\dfrac{\mathrm{d}f}{\mathrm{d}\eta}}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}\eta} = \frac{1}{\beta^2}\frac{\mathrm{d}^2 g}{\mathrm{d}\xi^2}, \quad \frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} = \frac{\mathrm{d}\dfrac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}\eta} = \frac{1}{\beta^3}\frac{\mathrm{d}^3 g}{\mathrm{d}\xi^3} \right. \tag{20}$$

Where the momentum eq. (13) and the boundary conditions are converted to:

$$g''' = \beta^{2n-1}(-g'')^{1-n}\left[\beta S\left(g' + \frac{1-n}{1+n}\xi g'' + \mathrm{H}g'\right) + g'^2 - \frac{2n}{n+1}gg''\right] \tag{21}$$

$$g(0) = 0, \qquad g'(0) = \beta, \qquad g(1) = \frac{2-n}{2n}\beta S, \qquad g''(1) = 0 \tag{22}$$

When $n \geq 0.5$, we can obtain that $\beta \to 0 \Rightarrow g''' \equiv 0(\xi \in [0, 1])$. Also, we can obtain:

$$\begin{cases} g''' \equiv 0(\xi \in [0, 1]) \\ g''(1) = 0 \end{cases} \Rightarrow g'' \equiv 0(\xi \in [0, 1]) \Rightarrow g' \equiv g'(0) = \beta \Rightarrow g(\xi) = \beta\xi + g(0) \tag{23}$$

$$\begin{cases} g(\xi) = \beta\xi + g(0) \\ g(0) = 0 \end{cases} \Rightarrow g(\xi) = \beta\xi \Rightarrow f(\eta) = \eta \tag{24}$$

$$\begin{cases} f(\eta) = \eta \\ f(\beta) = \dfrac{2-n}{2n}\beta S \end{cases} \Rightarrow \frac{2-n}{2n}S = 1 \Rightarrow S = \frac{2n}{2-n} \tag{25}$$

So, the critical value is also $S_0 = 2n/(2-n)$ which indicates that the magnetic field has no effect on critical value. In other word, the critical value is the same for different values of the Hartmann number.

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