DIFFUSION MODELS WITH WEAKLY SINGULAR KERNELS IN THE FADING MEMORIES
How the Integral-Balance Method Can Be Applied?

by

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This work presents an attempt to apply the integral-balance approach to diffusion models with fading memories expressed by weakly singular kernels. It demonstrates how three integration techniques (heat-balance integral method, double-integration method, and frozen front approach) work with a general parabolic profile with unspecified exponent and result in closed-form solutions. The main steps are exemplified by solutions where the fading memory is represented by Volterra integrals and by a time-fractional Riemann-Liouville derivative.

Key words: *heat-balance integral method, double-integration method, frozen front approach, weakly singular kernel, fading memory, fractional derivative, approximate solution*

Introduction

This paper reports results on application of the integral-balance method to diffusion equations with memory terms expressed with weakly singular (power-law) kernels. The method is explained by solution of examples where the memory is expressed as Volterra integrals and time-fractional Riemann-Liouville derivative. The solutions are straightforward and address three versions of the integral-balance method resulting in closed-form approximate profiles.

The diffusion with a fading memory

Diffusion phenomena, of heat or mass, are generally described as a rate law [1] $\partial u/\partial t = -\partial J/\partial x$. With the assumption that the flux $J(x, t)$ is proportional to the gradient (1b), *i. e.* $J(x, t) = -D\partial u(x, t)/\partial x$ we get:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Thus, defining the diffusivity D as a transport coefficient we get the classical Fick (Fourier) eq. (1) with unphysical infinite speed of propagation of the flux. The latter invokes for a damping function relating the model to the real processes in order to correct the unphysical eq. (1). To be precise in our comments, we address problems concerning diffusion in short-times where the relaxation takes place; then at large times we may use eq. (1) correctly. In the diffu-

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sion of heat, the damping function was conceived by Cattaneo [2] through a Volterra type integral where the flux J and the relaxation (damping) function $R(x, t)$ are assumed to satisfy the constitutive equation [3] $J(x, t) = -\int_0^\infty R(x,t)\nabla u(x, t-\tau)\mathrm{d}\tau$.

In homogeneous media, the damping function should be time-dependent and independent of the space co-ordinate. The present communication refers to a relaxation function expressed as a power-law kernel $R(x, t) = (t-\tau)^{-\mu}$. This leads to eq, (2a):

$$\frac{\partial u}{\partial t} = a_e \int_0^t \frac{1}{(t-\tau)^\mu} \frac{\partial^2 u(x,t)}{\partial x^2} \mathrm{d}\tau \Rightarrow \frac{1}{\Gamma(\mu)} \frac{\partial u}{\partial t} = a_e^\mu I_t^\mu \left[\frac{\partial^2 u(x,\tau)}{\partial x^2} \right], \quad 0 < \mu < 1 \qquad \text{(2a,b)}$$

The fractional order is $0 < \mu \leq 1$ [4] and the elastic diffusivity a_e has a dimension $[\mathrm{m}^2/\mathrm{s}^{2-\mu}]$. The right-hand side of eq. (2a) can be presented as a fractional integral of the Riemann-Liouville sense [5] which allows to obtain the form expressed by eq. (2b). This approach is commonly applied for modelling flows of viscoelastic fluids [6-8], granular flow [9] or heat conduction with memory [10].

The diffusion equations with fading memories represented by Volterra integrals are solved mainly by approximate analytical [11] or numerical methods [7, 11, 12]. The approximate solutions developed in this work utilize the Integral-Balance approach belonging to the family of the weighted residuals methods [13] in two of its basic versions: (1) The Heat-Balance Integral Method (HBIM) and (2) Double-Integration Method (DIM). Additionally, a semi-empirical method termed Frozen Front Approach (FFA) [14-17] was successfully applied to equations where the relaxation function is presented by fractional-time derivatives such as transient flow models of viscoelastic fluids [18, 19]. The results of FFA are compared in this work to the solutions developed by HBIM and DIM.

The Integral-balance method

The integral balance method [20] suggests a finite depth of penetration $\delta(t)$ of the diffusant (heat or mass), which means a sharp front at $x = \delta$. The Goodman's boundary conditions are [20]: $u(\delta, t) = 0$ and $u_x(\delta, t) = 0$. The distribution within the penetration layer can be approximated by an assumed profile $u_a(x, \delta)$.

HBIM

The core of HBIM is the integration of both sides of the governing equation from 0 to δ. Taking as example eq. (1) this approach provides:

$$\int_0^\delta \frac{\partial u}{\partial t} \mathrm{d}x = \int_0^\delta D \frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} \int_0^\delta u \mathrm{d}x = -D \frac{\partial u}{\partial} \bigg|_{x=0} \qquad \text{(3a, b)}$$

The integral (3b) is termed heat-balance integral [20]. This approach implies that the function $u(x, t)$ satisfies the governing equation in average. Replacement of the function $u(x, t)$ by $u_a(x, \delta)$ in (3b) yields an equation describing the time-evolution of the penetration depth $\delta(t)$.

DTM

The main disadvantage of HBIM is that the gradient at x = 0 (see 3b) should be approximated through the assumed profile. The first step of DIM is the integration from 0 to x. The second step is the integration again from 0 to δ. Especially to eq. (1) and the intermediate result (3b) the final equation of DIM is [21]:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\delta \left(\int_0^x u \mathrm{d}x \right) \mathrm{d}x = Du(0,t) \qquad (4)$$

This method of solution is applied in the present work in a slightly different form and the details are given in the following examples.

The method through examples

In the following examples, the assumed profile is $u_a = b_1 + b_2(1 - x/b_3)^n$ where $n > 0$ is undefined exponent [22]. Applying the Goodman' boundary conditions, for any $n > 0$ we read the approximate profile as $u_a = u_0(1 - x/\delta)^n$. For seek of simplicity we assume $u_0 = 1$ (Dirichlet problem), which is equivalent to use of the dimensionless profile $\widetilde{u}_a = u_a/u_0 = (1 - x/\delta)^n$.

Example 1: Simple equation with elastic effect only

We start with a simple parabolic equation modelling rigid heat conductors and relaxing viscoelasticity [7] with fading memory (expressed by the kernel $(t-s)^{-\mu}$ with a dummy (time delay) (variable τ) and effective elastic diffusivity $a_e[\mathrm{m}^2/\mathrm{s}^{-\mu}]$. Hence, the Dirichlet problem with initial and boundary conditions is formulated as:

$$\frac{\partial u}{\partial t} = a_e \int_0^t \frac{1}{(t-\tau)^\mu} \frac{\partial^2 u(x,\tau)}{\partial x^2} \mathrm{d}\tau, \quad 0 < \mu < 1 \tag{5a}$$

$$u(x,0) = 0, \quad 0 \le x \le \infty, \quad u(x,t) = 0, \quad x \to \infty \quad \text{and} \quad u(0,t) = u_0 \tag{6b, c, d}$$

Because the problem (5-6) is parabolic [7], then the finite penetration depth $\delta(t)$ is an *ad hoc* correction of the basic inadequacy related to infinite speed inherent of the model (1) as explained. With this concept, we suggest a finite penetration depth δ replacing the condition (6b, c) by $u(\delta, t) = 0$ and $u_x(\delta, t) = 0$.

HBIM solution

Applying the HBIM approach to (5a), with the Leibniz rule to the left-side and replacing the function u by u_a we read:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta u_a(x,\delta)\mathrm{d}x = \int_0^\delta \left[a_e \int_0^t \frac{1}{(t-\tau)^\mu} \frac{\partial^2 u_a(x,\delta)}{\partial x^2}\mathrm{d}\tau \right]\mathrm{d}x \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta}{n+1}\right) = a_e \int_0^t \frac{1}{(t-\tau)^\mu}\left(\frac{n}{\delta}\right)\mathrm{d}\tau \tag{7a, b}$$

The integration of (7b) again from 0 to δ yields:

$$\frac{1}{2(n+1)}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2 = a_e n \int_0^t \frac{1}{(t-\tau)^\mu}\mathrm{d}\tau \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\delta^2 = a_e 2n(n+1)\frac{t^{1-\mu}}{1-\mu} \tag{8a, b}$$

The integration of (8b) with the initial condition $\delta(t = 0)$ yields:

$$\delta^2 = a_e 2n(n+1)\frac{t^{2-\mu}}{(1-\mu)(2-\mu)} \Rightarrow \delta = \sqrt{a_e}\sqrt{t^{2-\mu}}\sqrt{\frac{2n(n+1)}{(1-\mu)(2-\mu)}} \tag{9a, b}$$

DIM solution

The first integration of eq. (5a) from 0 to x yields:

$$\int_0^x \frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}x = \int_0^x \left[a_e \frac{1}{(t-\tau)^\mu} \frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau \right]\mathrm{d}x \tag{10}$$

Further, the integral from 0 to δ, however, can be presented as $\int_0^\delta f(\bullet)\mathrm{d}x = \int_0^x f(\bullet)\mathrm{d}x + \int_x^\delta f(\bullet)\mathrm{d}x$. Therefore, eq. (7a) reads:

$$\int_0^x \frac{\partial u}{\partial t}\mathrm{d}x + \int_x^\delta \frac{\partial u}{\partial t}\mathrm{d}x = \int_0^x \left[a_\mathrm{e}\int_0^t \frac{1}{(t-\tau)^\mu}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau \right]\mathrm{d}x + \int_0^\delta \left[a_\mathrm{e}\int_0^t \frac{1}{(t-\tau)^\mu}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau \right]\mathrm{d}x \quad (11)$$

Now, subtracting (10) from (11) we obtain:

$$\int_x^\delta \frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}x = \int_x^\delta \left[a_\mathrm{e}\int_0^t \frac{1}{(t-\tau)^\mu}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau \right]\mathrm{d}x \quad (12)$$

Integrating (12) from 0 to δ we get the final equation of DIM, specific to the problem at issue:

$$\int_0^\delta \left(\int_x^\delta \frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}x \right)\mathrm{d}x = \int_0^\delta \left\{ \int_x^\delta \left[a_\mathrm{e}\int_0^t \frac{1}{(t-\tau)^\mu}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau \right]\mathrm{d}x \right\}\mathrm{d}x \quad (13)$$

Replacing u by the assumed parabolic profile u_a in eq. (13) and after the double integration we get:

$$\frac{1}{(n+1)(n+2)}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2 = a_\mathrm{e}\int_0^t \frac{1}{(t-\tau)^\mu}\mathrm{d}\tau \quad (14)$$

Since the right side of (14) is a convolution integral of 1 and t^μ we have:

$$\frac{\mathrm{d}\delta^2}{\mathrm{d}t} = a_\mathrm{e}(n+1)(n+2)\frac{t^{1-\mu}}{1-\mu}, \quad \delta^2 = a_e(n+1)(n+2)\frac{t^{2-\mu}}{(2-\mu)(1-\mu)} \quad (15\text{a, b})$$

Finally, the DIM solution about the penetration depth is:

$$\delta_\mathrm{DIM} = \sqrt{a_\mathrm{e}}\sqrt{t^{2-\mu}}\sqrt{\frac{(n+1)(n+2)}{(1-\mu)(2-\mu)}}, \quad 0<\mu<1 \quad (16)$$

Frozen front approach (FFA)

This approach has been applied to the first integral balance (HBIM) solutions of the fractional diffusion equation [15-17]. It was more physically motivated rather than mathematically proved. The main physical assumption behind the FFA is based on the assumption that the rate of the penetration front is extremely slow, that is for a certain time range the value of δ could be considered as a constant. However, this assumption is valid only in the evaluation of the convolution integral. The comments after the solution of the example by FFA explain why we may separate the penetration depth concept and the delay expressed by the Volterra integral.

The memory integral in (5a) is a time-convolution between the memory function $f_\mathrm{M}(t) = t^{-\mu}$ and $u_{xx}(x, t)$, thus the integration in (5a) is performed with respect to the dummy variable τ, while the time t is a parameter. With the assumption of "frozen" δ the approximate profile does not depend on τ and consequently with the general form of the memory kernel in $(t-s)^{-\mu}$ and the assumed profile $\widetilde{u}_\mathrm{a} = u_\mathrm{a}/u_0 = (1 - x/\delta)^\mathrm{n}$ we read:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta_\mu}{n+1}\right) = \int_0^\delta \left[a_\mathrm{e}\frac{n(n-1)}{\delta_\mu^2}\left(1-\frac{x}{\delta_\mu}\right)^{n-2}\int_0^t \frac{1}{(t-\tau)^\mu}\mathrm{d}\tau \right]\mathrm{d}x \quad (17)$$

The integration of eq.(17) (used in [15, 17, 19]) (with the initial condition is $\delta\mu(t=0) = 0$) yields:

$$\frac{\mathrm{d}\delta_\mu^2}{\mathrm{d}t} = 2a_\mathrm{e}n(n+1)\frac{1}{1-\mu}t^{1-\mu} \Rightarrow \delta_\mu = \sqrt{a_\mathrm{e}t^{2-\mu}}\sqrt{\frac{2n(n+1)}{(1-\mu)(2-\mu)}}, \quad 0<\mu<1 \quad (18\text{a, b})$$

The result (18b) is the same as that obtained by the HBIM solution. In these two examples, we present the penetration depth as a time dependent function proportional to the term

$[2n(n+1)]^{1/2}$. This was especially done, because in the viscous case with $\mu = 1$ the penetration depth is $\delta(t)_{m=1} = (\nu t)^{1/2}[2n(n+1)]^{1/2}$ [22], where ν [m^{2}s^{-1}] is the viscous diffusivity of momentum.

The memory term in the parabolic eq. (3), in fact, accounts for a delay in the propagation speed of the disturbances imposed at the boundary $x = 0$, thus compensating the unphysical infinite speed in the classical diffusion eq. (1). The concept of a finite penetration depth $\delta(t)$, however, does the same, but this is valid for large times, while the weakly singular kernels accounts for short-time relaxation effects.

Therefore, applying the integral-balance concept to a model containing a memory term expressed as a Volterra integral, we overlap two techniques relevant to the concept of finite propagation speed. The FFA, however, separates the "objects", *i. e.* the terms of the model to which these two concepts work. That is, the concept of a final depth $\delta(t)$ works on all terms without memories, while in the convolution integral δ is assumed as a constant due to the short-time effect of this term. Discrimination between short and long-time effects exists in the construction of the model (3), for instance: the left side accounts for the long-time effects, while the right side represents the short-time relaxations. Such discrimination will be more obvious in Example 2 developed in the next section.

Example 2: A models with a memory kernel incorporated in a fractional-time derivative and a viscous damping

Here, we refer to the fact that the relaxation can be presented as a time-fractional derivative D_t^{μ} (see eq. 20). Modelling a transient flow of a second-grade viscoelastic fluid, as example, we have a governing equation [12, 15] which in a diffusion form can be expressed as:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + p D_t^{\mu} \frac{\partial^2 u}{\partial y^2} \tag{19}$$

Here the fractional derivative is described in the Riemann-Liouville sense [5], namely:

$$D_t^{\mu} f(x,t) = \frac{1}{\Gamma(1-\mu)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{1}{(t-\tau)^{\mu}} f(x,\tau)\mathrm{d}\tau \tag{20}$$

Further, $\nu = \eta/\rho$ [m^{2}s^{-1}] is the kinematic viscosity of the fluid, $p = \alpha_1/\rho$ [m^2] is the commonly expressed as $p = \nu\lambda_r$, where λ_r [s] is the relaxation time, and α_1 – a viscoelastic parameter termed "the first normal stress modulus". The operator D_t^{μ} $[\rho y^2 \eta^{-1}]^{\mu}$ has a fractional time dimension, and η [Nm^{-2}s^{-1}] is the fluid dynamic viscosity. Now, we will perform solutions to the Stokes's first problem, that is with $u(0, t) = u_0$ and zero-initial condition $u(x, t) = 0$, $0 \le x \le \infty$, by the three integration techniques used in this work.

HBIM solution

Integrating eq. (19) from 0 to δ we get:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{\delta} u \mathrm{d}x = -\nu \left(\frac{\partial u}{\partial x} \right)_{x=0} + \int_0^{\delta} \left(p D_t^{\mu} \frac{\partial^2 u}{\partial x^2} \right) \mathrm{d}x \tag{21}$$

with the assumed profile $\widetilde{u}_a = u_a/u_0 = (1 - x/\delta)^n$ the integration in (22) yields:

$$\frac{1}{n+1} \frac{\mathrm{d}}{\mathrm{d}t} \delta = \nu \frac{n}{\delta} + p \int_0^{\delta} \left[\frac{1}{\Gamma(1-\mu)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{n}{\delta} \frac{1}{(t-\tau)} \mathrm{d}\tau \right] \mathrm{d}x \tag{22}$$

The new integration of eq. (22) from 0 to δ results in:

$$\frac{1}{2(n+1)}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2 = \nu n + np\frac{1}{\Gamma(1-\mu)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{1}{(1-\tau)}\mathrm{d}\tau \Rightarrow$$
$$\Rightarrow \frac{\mathrm{d}\delta^2}{\mathrm{d}t} = \nu 2n(n+1) + 2pm(n+1)\frac{t^{-\mu}}{\Gamma(1-\mu)} \qquad \text{(23a, b)}$$

The last term in (23a) is a Riemann-Liouville fractional derivative of 1, thus we get a simple equations about expressed by (23b). From (23b), with the initial condition $\delta(t = 0) = 0$, we get:

$$\delta^2 = 2n(n+1)\left[\nu t + p\frac{t^{1-\mu}}{(1-\mu)\Gamma(1-\mu)}\right] \Rightarrow \delta = \sqrt{\nu t}\sqrt{2n(n+1)\left[1+\left(\frac{p}{\nu t^{\mu}}\right)\frac{1}{\Gamma(2-\mu)}\right]} \qquad \text{(24a, b)}$$

From (24b) with $p = 0$ we obtain the classical HBIM solution for viscous flow (First Stokes' problem) with $\delta_\nu(t) = (\nu t)^{1/2}[2n(n + 1)]^{1/2}$, while for $\nu = 0$ from eq. (24a) we have the elastic solution.

DIM solution

Integrating eq. (19) first from 0 to x and then from 0 to δ we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta\int_0^x xu\mathrm{d}x = \nu u(0,t) + \int_0^\delta\int_0^x\left(pD_t^\mu\frac{\partial^2 u}{\partial x^2}\right)\mathrm{d}x \qquad (25)$$

with the assumed profile $\tilde{u}_a = u_a/u_0 = (1 - x/\delta)^n$ and after the integration of eq. (25) the result is:

$$\frac{1}{(n+1)(n+2)}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2 = \nu + p\frac{1}{\Gamma(1-\mu)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{1}{(t-\tau)^\nu}\mathrm{d}\tau \qquad (26)$$

The last term in eq. (25) is the same as that obtained in eq. (23a), that is, we obtain:

$$\frac{\mathrm{d}\delta^2}{\mathrm{d}t} = (n+1)(n+2)\left[\nu t + p\frac{t^{1-\mu}}{\Gamma(1-\mu)}\right], \quad \delta_{\mathrm{DIM}} = \sqrt{\nu t}\sqrt{(n+1)(n+2)\left[1+\left(\frac{p}{\nu t^\mu}\right)\frac{1}{\Gamma(2-\mu)}\right]} \qquad \text{(27a, b)}$$

FFA solution

The integration of eq. (19) over the penetration depth δ_{FFA} with the profile $\tilde{u}_a = u_a/u_0 = (1 - y/\delta_{\mathrm{FFA}})^n$ and $\nu = 0$ results in an equation about the elastic penetration depth δ_{FFA}^e:

$$\frac{\mathrm{d}(\delta_{FFA}^e)^2}{\mathrm{d}t} = 2p\frac{n(n+1)}{(1-\mu)\Gamma(1-\mu)}t^{-\mu} \Rightarrow \delta_{FFA}^e = \sqrt{2pn(n+1)t\left(\frac{j_\mu}{t^\mu}\right)}, \quad j_\mu = \frac{1}{(1-\mu)\Gamma(1-\mu)} \qquad \text{(28a,b,c)}$$

If the complete eq. (19) is considered, then the HBIM result [19] is:

$$\delta_{\mathrm{FFA}}^{\nu+e} = \sqrt{\nu t}\sqrt{2n(n+1)\left(1+\frac{p}{\nu t^\mu}j_\mu\right)} \qquad (29)$$

The approximate profiles and relevant comments

The three integration techniques applied to eq. (7a) provided different expressions about $\delta(t)$, which are expectable results. The HBIM solutions of Example 2, for instance, lead to following approximate profiles:

$$u_{\mathrm{a(elastic)}} = \left(1 - \frac{x}{\sqrt{pt}\sqrt{2n(n+1)}\sqrt{\dfrac{j_\mu}{t^\mu}}}\right)^n \quad \text{and}$$

$$u_{\mathrm{a(viscous\ and\ elastic)}} = \left(1 - \frac{x}{\sqrt{pt}\sqrt{2n(n+1)\left[1 + \dfrac{p}{\nu t^\mu} j_\mu\right]}}\right)^n \tag{30a, b}$$

The profiles (30a, b) are correct because the assumed profile $\widetilde{u}_{\mathrm{a}} = u_{\mathrm{a}}/u_0 = (1 - y/\delta)^n$ satisfies all boundary conditions imposed at the boundaries of the penetration layer δ for any $n > 1$. The developed integral-balance solutions, however, do not specify the value of the exponent n. Therefore, the minimization of the error of approximation, a procedure requiring the approximate profile to satisfy the original equation (eq. 5a or eq. 19) is the next step of the complete solution; a step defining the optimal exponent by application of additional constraints to the developed approximate solutions.

Penetration depth ratio, the Deborah number and the reasons to apply FFA

In eq. (30), as well as in eq. (28b), the fading memory is presented by the decaying ratio $\mathrm{De} = p/\nu t^\mu = (p/\nu)t^{-\mu} = \lambda_{\mathrm{r}}/t^\mu$, where $\mathrm{De} = p/\nu t^\mu$ is the Deborah number [19, 23]. The ratio of the elastic to the viscous penetration depths is $\Delta_{\mathrm{e}-\nu} = \delta^{\mathrm{e}}/\delta^\nu = (\mathrm{De})^{1/2}(j_\mu)^{1/2}$. Furthermore, the elastic penetration depth is proportional to $p^{1/2}$ (modelled by the term with memory) while the viscous one (modelled by terms without memory) is proportional to $\nu^{1/2}$. The typical relaxation time λ_{r} for viscoelastic fluids is of order $(1 - 10)\cdot 10^{-3}$ s for a Newtonian dynamic viscosity (η) range from 2 to $10\cdot 10^{-3}$ N/m^{2}s [24], that is a kinematic viscosity ν of order 10^{-6} m^2/s. Hence, simply the ratio of the speeds (and the lengths, too) of both penetration depths is proportional to $p^{1/2}/\nu^{1/2} = (\alpha_1)^{1/2}/\eta = (\mathrm{De}t^\mu)^{1/2}$ having order of magnitude of about $O(10^2)$.

The above comments figure the physical basis of FFA, but the principle questions are: (1) What really we did mathematically inexact with the FFA solution? (2) What is the secret of the FFA solution? The answers are straightforward: nothing incorrect was done and nothing secret was used. We will explain this answer briefly. In the correct HBIM solution we integrate twice (from 0 to δ) thus making the convolution integral independent of δ. Practically it does not matter how the double integration with respect to the space co-ordinate x is performed, *i. e.* when the assumed profile is inside or outside the convolution integral. Finally, the results from HBIM and FFA are identical. With the DIM solution, the effect of the double integration is obvious, *i. e.* the convolution integral becomes independent of the assumed profile. More details about FFA and comparative analyses including mathematically correct solutions with HBIM and DIM are available elsewhere [25].

The optimal exponents of the approximate profiles

Error measure and optimal exponent

The quest for optimal exponent of the approximate profile means that it should satisfy the original diffusion model for any point of the domain $0 \le x \le \delta(t)$. In case of the model (3), it follows that for any point of the domain $0 \le x \le \delta(t)$ the residual function $\varphi_\mu(\mu, n)$ expressed by eq. (31) should be minimal:

$$\varphi_\mu(\mu, n) = \frac{\partial u_a}{\partial t} - a_e \int_0^t (t-\tau)^{-\mu} \frac{\partial^2 u_a(x,\tau)}{\partial x^2} \to \min \tag{31}$$

Therefore, we need $E_{\mu 1}(n, t) = \int_0^\delta \varphi_\mu(\mu, n)\mathrm{d}x \to \min$. The expression of $E_{\mu 1}(n, t)$ for the HBIM solution of model (3) with $\delta \mathrm{d}\delta/\mathrm{d}t = (1/2)\mathrm{d}\delta^2/\mathrm{d}t = a_e t^{1-\mu} 2n(n+1)(1-\mu)^{-1}$ is:

$$E_{\mu 1}(n,t) = \frac{\sqrt{a_e}}{\sqrt{t^\mu}} e_{\mu 1}(\mu, n), \quad e_{\mu 1}(\mu, n) = \frac{2-\mu}{1-\mu} \frac{2-n}{2n^2(n+1)^2} \tag{32a,b}$$

That is, $E_{\mu 1}(n, t)$ decays in time and we have to minimize $e_{\mu 1}(\mu, n)$ with respect to n.The trivial root of (32b) is $n = 2$ for any μ, but we have to bear in mind that $e_{\mu 1}(n, t)$ is a scaled function and we should look for it is minimum. If we impose the condition $E_{\mu 1}(n, t) > 0$, it should require $n < 2$, but this condition is optional. The optimization procedure performed by Maple results in optimal $n_{\mathrm{opt}} = 2.808$. The error function $e_{\mu 1}(\mu, n)$ varies from $-7.459 \cdot 10^{-3}$ for $\mu = 0.1$ to $-3.886 \cdot 10^{-3}$ for $\mu = 0.9$. With $n_{\mathrm{opt}} = 2.808$ we have $E_{\mu 1}(n, t) < 0$ for all $0 < \mu < 1$ that is the approximate solution underestimates the exact one. Otherwise, in order to avoid any doubts about the integration in $E_{\mu 1}(n, t)$ we may integrate it twice from 0 to δ, thus making the convolution integral independent of δ. The results of this step is:

$$\int_0^\delta E_{\mu 1}(n,t)\mathrm{d}x = a_e \frac{t^{1-\mu}}{1-\mu}[n(n+1)-1] \tag{33}$$

The trivial positive solution of eq. (33) is $n = -1/2 + (5)^{1/2}/2 \approx 0.618$. However, this approach does not provide adequate value of the exponent, because it is lower than 1 as well as it is lower that the exponent of the single viscous spreading of the profile as it will be demonstrated next.

Obviously, the simple approach by either eqs. (32) or (33), to determine the optimal exponents does not lead to desired and physically adequate results. Therefore, the more correct approach is to minimize the squared-error function over the penetration depth, that is $E_{\mu 2}(n, t) = \int_0^\delta [\varphi_\mu(\mu, n)]^2 \mathrm{d}x \to \min$, also known as the Langford criterion applied to HBIM solution [26]). With the complete model (20) and the FFA solution it was established [19] that $n \approx 1/(2\mathrm{De})^{1/2}$, which practically meaning a time-dependent exponent. Since in absence of elastic effects, *i. e.* for $p = 0$ we obtain the classical diffusion model, then the HBIM solution has an optimal exponent $n \approx 2.35$ [21]. The optimal exponent established for $p = 0$ should be considered as the lower limit of exponents modelling the elastic effects only. Simple calculations with $n = 2$ and $n = 3$ provide $\mathrm{De} \approx 0.125$ and $\mathrm{De} \approx 0.055$, respectively. Further, for the optimal Newtonian profile ($p = 0$) with exponent $n \approx 2.35$ we get $\mathrm{De} \approx 0.100$. The typical relaxation time λ_r for viscoelastic fluids is in the range $(1\text{-}10) \cdot 10^{-3}$ s for a Newtonian viscosity range from 2 to $10 \cdot 10^{-3}$ N/m^{2}s [24]. Thus, for observation time of about 1*s*, for instance, we get Deborah numbers in the range $(2\text{-}10) \cdot 10^{-3}$ and from $n \approx 1/(2\mathrm{De})^{1/2}$ we get a range of optimal exponents from $n \approx 15.81$ to $n \approx 7.071$, respectively. Hence, the lower Deborah numbers, the higher exponents of the parabolic profile for the short-time (with dominating elastic effects) solutions. Examples related to the Stokes' first problem with the complete model (20) are available elsewhere [19]. The following example demonstrates how these approximations work for the case of $\mu = 1/2$ and elastic effects only.

A comparative example for μ = 1/2 and elastic effects only

Let us consider the simple case of the model (3) for $\mu = 1/2$. Applying the Laplace transform in the time domain to (3) we get $u_{xx}(s) - [s(s)^{1/2}/(\pi)^{1/2}]u(s)$. Further, expressing the solution with respect to x we get $u(s) = \exp[-xs(s)^{1/2}/(\pi)^{1/2}$. The approximation of $u(s)$ as a series with respect to s and a consequent truncation to the first two terms allow the solution to be expressed approximately as $u_z(x, t) = u_z(z) \approx 1 - \exp[-z/(\pi)^{1/2}]$; here $z = x/t^{3/2}$ is the similarity variable $z = = x/(t^{2-\mu})^{1/2}$ for $\mu = 1/2$. The plots in figs. 1 and 2 illustrate this almost exact solution compared to that developed by FFA (the same as HBIM) for two different values of the exponent n.

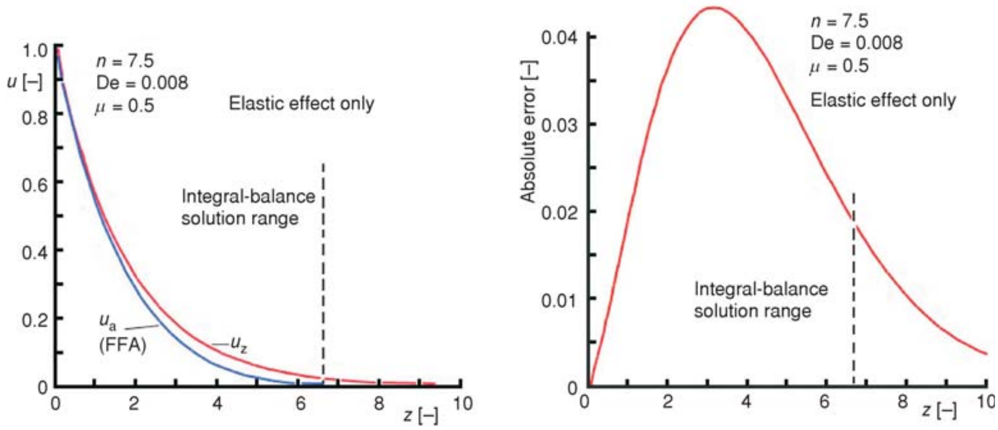

Figure 1. Solution of the elastic model (3) for n = 7.5; left: comparison of the approximate (FFA) integral-balance solution and the almost exact u_z solution, right: absolute errors

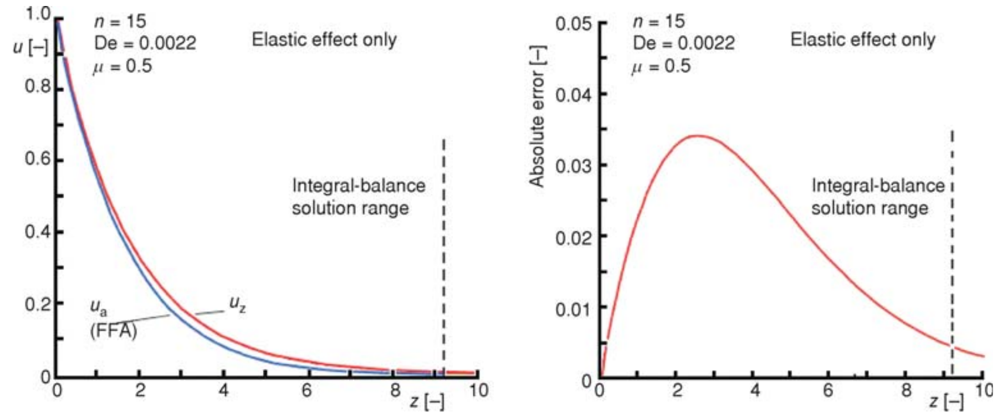

Figure 2. Solution of the elastic model (3) for n = 15; left: comparison of the approximate (FFA) integral-balance solution and the almost exact u_z solution, right: absolute errors

The increase in the value of the exponent n, that is the decrease in the Deborah number leads to better approximations of the elastic profiles and lower absolute errors. These examples reveal acceptable absolute errors of approximation in contrast to those in fig. 3 where the exponent of the profile is close to the viscous limit $n = 2.35$.

Conclusions

This work presented an attempt to apply the integral-balance method to diffusion models with fading memories with weakly singular kernels. It was shown how the approach works

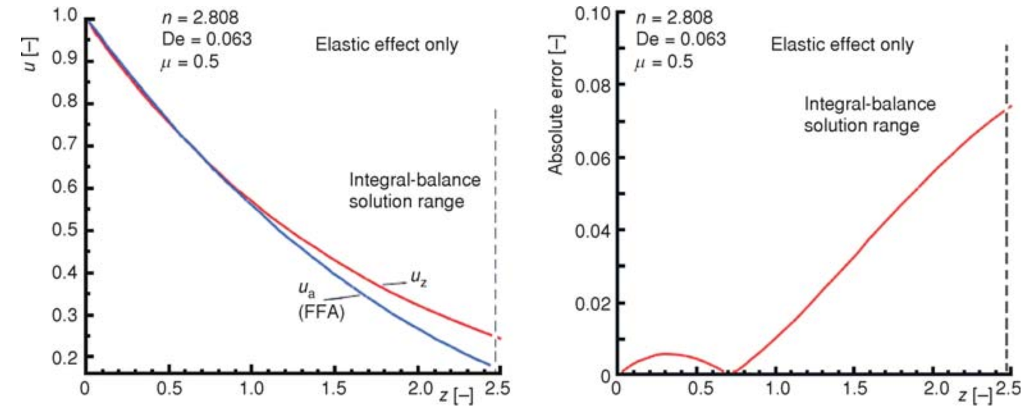

Figure 3. Solution of the elastic model (3) for n = 2.808; left: comparison of the approximate (FFA) integral-balance solution and the almost exact u_z solution, right: absolute errors

with fading terms as Volterra integrals and assumed general parabolic profile with unspecified exponent. Three integration techniques of the integral-balance method have been successfully demonstrated through solutions of examples. In general, this is a new implementation of an old idea and related methods to new models and we hope the demonstrated technique could be useful in solutions of other problems.

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