VARIATIONAL FORMULATIONS FOR SOLITON EQUATIONS ARISING IN WATER TRANSPORT IN POROUS SOILS

by

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The semi-inverse method is adopted to establish variational principles for Korteweg De-Vries-like equations arising in water transport in porous soils.

Key words: *semi-inverse method, integrable inhomogeneous equations*

Introduction

In this paper, the following two kinds of integrable inhomogeneous soliton equations are studied [1]:

$$u_t = \left(2u^3 + u_{xx} + \frac{x\lambda_t u}{\lambda} + f(t)u\right)_x \tag{1}$$

and

$$\begin{cases} u_t - u_{xx} + 2u^2 v - 2ku - (hu)_x = 0 \\ v_t + v_{xx} - 2uv^2 + 2kv - (hv)_x = 0 \end{cases} \tag{2}$$

where $f(t)$ and $h = h(x, t)$ are arbitrary functions and λ is a spectral parameter in Lax pairs.

Equations (1) and (2) can describe, respectively, 1-D and 2-D water transport in porous soils, and the solitary solutions can pick out the main property of the contaminated land. In this paper, we adopt the semi-inverse method [2] to search for the variational formulations for the above equations.

Variational formulation

Variational methods [3, 4] have been popular tools for non-linear analysis. The semi-inverse method proposed by He [2] is proved to be efficient to search for variational principles.

According to eq. (1), we can introduce a special functional ϕ defined as:

$$\phi_x = u \quad \text{and} \quad \phi_t = 2u^3 + u_{xx} + \frac{x\lambda_t u}{\lambda} + f(t)u \tag{3}$$

Construct a trial-functional in the form [2]:

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$$J = \iint L \mathrm{d}t\mathrm{d}x = \iint \left\{ u\phi_t - \left[2u^3 + u_{xx} + \frac{x\lambda_t u}{\lambda} + f(t)u \right]\phi_x + F \right\} \mathrm{d}t\mathrm{d}x \tag{4}$$

where L denotes Lagrange's function, and $F = F(u, u_x, u_t, u_{xx}, \ ... \ )$.

The advantage of the above trial-functional is that the Euler-Lagrange equation with respect to ϕ is eq. (3). The stationary condition with respect to u reads;

$$\phi_t - 6u^2\phi_x - \phi_{xxx} - \frac{x\lambda_t}{\lambda}\phi_x - f(t)\phi_x + \frac{\delta F}{\delta u} = 0 \tag{5}$$

where $\delta L/\delta u$ is the variational derivative defined as [2]:

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_x} - \frac{\partial}{\partial t}\frac{\partial F}{\partial u_t} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right) + \frac{\partial^2}{\partial xt}\left(\frac{\partial F}{\partial u_{xx}}\right) + \cdots \tag{6}$$

By eq. (3), eq. (5) becomes:

$$\frac{\delta F}{\delta u} = 4u^3 \tag{7}$$

from which F can be identified, which is $F = u^4$.

$$J = \iint L \mathrm{d}t\mathrm{d}x = \iint \left\{ u\phi_t - \left[2u^3 + u_{xx} + \frac{x\lambda_t u}{\lambda} + f(t)u \right]\phi_x + 4u^4 \right\} \mathrm{d}t\mathrm{d}x \tag{8}$$

For the system, eq. (2), we can construct a trial-Lagrange function in the form:

$$L = vu_t - u_{xx}v + u^2v^2 - 2kuv - (hu)_x v + F \tag{9}$$

The Euler-Lagrange equations with respect to v and u are the first equation of eq. (2), and the following one, respectively:

$$-v_t - v_{xx} + 2uv^2 - 2kv - h_x v + (hv)_x + \frac{\delta F}{\delta u} = 0 \tag{10}$$

Using the first equation of eq. (2), we can simplify eq. (10), which becomes:

$$\frac{\delta F}{\delta u} = h_x v \tag{11}$$

from which F can be determined as:

$$F = h_x uv \tag{12}$$

Therefore, we obtain the following variational principle for eq. (2):

$$J = \iint (vu_t - u_{xx}v + u^2v^2 - 2kuv - hu_x v)\mathrm{d}t\mathrm{d}x \tag{13}$$

Conclusions

The variational principle is important to elucidate main property of the problem. The energy for 1-D and 2-D water transport in porous soils depends mainly on u^4 and u^2v^2, respectively. These terms reveals that the moving wetted front or the velocity of the contaminated land depend upon u^4 for 1-D case, and u^2v^2 for 2-D case.

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