GDTM-PADÉ TECHNIQUE FOR THE NON-LINEAR DIFFERENTIAL-DIFFERENCE EQUATION

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Original scientific paper
DOI: 10.2298/TSCI1305305L

This paper focuses on applying the GDTM-Padé technique to solve the non-linear differential-difference equation. The bell-shaped solitary wave solution of Belov-Chaltikian lattice equation is considered. Comparison between the approximate solutions and the exact ones shows that this technique is an efficient and attractive method for solving the differential-difference equations.

Key words: *generalized differential transform method, Padé approximation, Belov-Chaltikian lattice*

Introduction

Differential-difference equations arise in many fields, such as air permeability in fabrics [1], inverse heat conduction problem [2], nanoscale heat transfer [3], nanoscale hydrodynamics [4], thermal excitation [5], and others [6, 7]. In this paper we will study the Belov-Chaltikian lattice equation, which is [8]:

$$\begin{aligned}\frac{\partial u_n}{\partial t} &= u_n(u_{n+1}-u_{n-1})-v_n+v_{n-1}\\ \frac{\partial v_n}{\partial t} &= v_n(u_{n+2}-u_{n-1})\end{aligned} \tag{1}$$

Equation (1) is originally introduced in the study of lattice analogues of W-algebras and has many applications in the fluid, plasmas, crystal lattice and so on [9, 10].

There are many analytical approaches to differential-difference equations, such as the Adomian decomposition method [11], the Exp-function method [12], the G'/G-expansion method [13], and the ADM-Padé technique [14]. In this paper the GDTM-Padé technique is adopted for solving the initial value problems of eq. (1). The GDTM-Padé technique is a combination of the generalized differential transform method [15] and the Padé technique [16]. As shown in [17, 18], the differential transform method can efficiently obtain analytical solutions of the non-linear equations in the form of a polynomial. It is different from the traditional Taylor series method, which requires the derivatives of the specified functions, resulting in expensive computation for high orders. Furthermore, the GDTM does not involve the symbolic computation of the integral and the perturbation technique. However, the approximate solution given by the GDTM may converge in a limited interval. To improve the convergence and the accuracy, the Padé approximation is applied to modify the approximate solution obtained by the GDTM.

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The GDTM-Padé technique

To illustrate the basic idea of the GDTM-Padé technique, we consider the general non-linear difference-differential equation:

$$N[u_n(t), u_{n+1}(t), u_{n-1}(t), u_{n+2}(t), \cdots] = 0.$$

where N is a non-linear differential operator, $u_n(t)$ is the unknown function with respect to the discrete spatial variable n and the temporal variable t.

Applying the 1-D differential transform method (GDTM), the differential transform of the k^{th} derivative of the function $u_n(t)$ is defined by:

$$U_n(k) = U(n,k) = \frac{1}{k!}\frac{\mathrm{d}^k u_n(t)}{\mathrm{d}t^k}\Big|_{t=t_0}$$

The differential inverse transform of $U_n(k)$ reads:

$$u_n(t) = \sum_{k=0}^{\infty} U_n(k)(t-t_0)^k = \sum_{k=0}^{\infty} U(n,k)(t-t_0)^k$$

Particularly, the function $u_n(t)$ can be formulated as a series when $t_0 = 0$:

$$u_n(t) = \sum_{k=0}^{\infty} U_n(k)t^k = \sum_{k=0}^{\infty} U(n,k)t^k$$

In practice, we can determine the coefficients $U(n,k)$ ($k = 1, \ldots, m$), and obtain the m^{th} order approximation of the function $u_n(t)$ given by:

$$u_{n,m}(t) = \sum_{k=0}^{m} U_n(k)t^k = \sum_{k=0}^{m} U(n,k)t^k \tag{2}$$

Table 1. The operations for generalized differential transform method

Original function	Transformed function
$f(n, t) = g(n, t) + h(n, t)$	$F(n, k) = G(n, k) + H(n, k)$
$f(n, t) = \alpha g(n, t)$	$F(n, k) = \alpha G(n, k)$
$f(n, t) = \partial g(n, t)/\partial t$	$F(n, k) = (k + 1)\, G(n,k + 1)$
$f(n, t) = g(n, t)\, h(n, t)$	$F(n,k) = \Sigma_{r=0}^{k} G(n,r)G(n,k-r)$
$f(n, t) = \partial^m g(n, t)/\partial t^m$	$F(n, k) = (k + 1) \ldots (k + m)\, G(n,k + m)$
$f(n, t) = g(n + s, t)$	$F(n, k) = G(n + s, k)$

The transformed operations for the GDTM are listed in tab. 1 [17, 18].

To improve the accuracy of the GDTM solution (2), the Padé approximation is used. For simplicity, we denote the $[L, M]$ Padé approximation to $f(x) = \Sigma_{k=0}^{\infty} a_k x^k$ by:

$$f[L, M] = P_L(x)/Q_m(x)$$

where $P_L(x) = p_0 + p_1x + p_2x^2 + \ldots + p_Lx^L$, and $Q_M(x) = 1 + q_1x + q_2x^2 + \ldots + q_Mx^M$ with the normalization condition $Q_M(0) = 1$. The coefficients of $P_L(x)$ and $Q_M(x)$ can be uniquely determined by comparing the first $L + M + 1$ terms of the functions $f[L, M]$ and $f(x)$. In the practical computation, the construction of the $[L, M]$ Padé approximation involves only algebra equations, which are solved by means of the Mathematica or Maple package. For simplicity, we call the solution obtained by the GDTM and the Padé approximation as the GDTM-Padé solution.

Numerical example

In this section, we test the Belov-Chaltikian lattice equation to verify the efficiency of the GDTM-Padé technique. We compare the performance of the GDTM-Padé technique with the original GDTM algorithm. All the numerical computations are performed by Mathematica 7.0.

Consider the initial value problem for the Belov-Chaltikian lattice eq. (1). We remark that the bell-shaped solitary wave solutions to eq. (1) are given by [8]:

$$u_n = \left[\ln \frac{g(n+0.5,t,z)}{g(n-0.5,t,z)} \right]_t$$

$$v_n = \frac{g(n+2.5,t,z)g(n-1.5,t,z)}{g(n+1.5,t,z)g(n-0.5,t,z)} \tag{3}$$

where $g(n,\ t,\ z) = 1 + e^{\eta}$ with $\eta = pn + qz + rt + \eta^0$, $r = \lambda^{-1}\ (e^p - 1)$, $q = \lambda^{-2}\ (e^{2p} - 1)$, $\lambda = (e^{1/2p} - e^{3/2p})/(e^{3/2p} - e^{-3/2p})$, p and η^0 are constant, and z is an auxiliary variable.

We suppose that the initial conditions to eq. (1) are defined by the above exact solutions (3) at $t = 0$, *i. e.*

$$u_{n0} = u_n(0) \text{ and } v_{n0} = v_n(0)$$

In this example, we set $p = 0.5I$, $z = 1$, and $\eta^0 = 0$.

Applying the GDTM, we can represent the transformed problem of eq. (1) in the recursive form:

$$(k+1)U(n,k+1) = \sum_{s=0}^{k} U(n,s)[U(n+1,k-s) - U(n-1,k-s)] - V(n,k) + V(n-1,k)$$

$$(k+1)V(n,k+1) = \sum_{s=0}^{k} V(n,s)[U(n+2,k-s) - U(n-1,k-s)] \tag{4}$$

The transformed initial conditions are $U(n, 0) = u_{n0}$, and $V(n, 0) = v_{n0}$.

Similarly, the implicit initial conditions can be constructed as $U(n-1, 0) = u_{n-1}(0)$, $U(n+1, 0) = u_{n+1}(0)$, $U(n+2, 0) = u_{n+2}(0)$, $V(n-1, 0) = v_{n-1}(0)$.

Based the recurrence (4), we can derive the coefficients $U(n, k)$ one by one, and obtain the approximate solution $u_{n,m}(t) = \Sigma_{k=0}^{m} U(n,k)t^k$. The approximation $v_{n,m}(t) = \Sigma_{k=0}^{m} V(n,k)t^k$ can be obtained similarly. The 6$^{\text{th}}$-order approximate solutions at $n = 4$ are given by:

$$u_{n,6} = -0.1148046602I - 0.0318913430t + 0.0263832285It^2 + 0.0086115113t^3$$
$$-0.0044383189It^4 - 0.001570717t^5 + 0.0006742217It^6 \tag{5}$$

$$v_{n,6} = 0.8128101475 + 0.0325816163It + 0.0380559556t^2 - 0.0085976478It^3$$
$$-0.0056223320t^4 + 0.0015280270It^5 + 0.0007627495t^6 \tag{6}$$

Applying the GDTM-Padé technique to the solution (5), we get the [3, 3] GDTM-Padé approximation:

$$u_n[3,3]=\frac{-0.1148046602I+0.0215791647t+0.0041445754I\,t^2-0.0011572965t^3}{1+0.4657520664I\,t+0.0643283483t^2+0.0398133041I\,t^3}$$

The [3, 3] GDTM-Padé approximation to $v_{n.6}$, eq. (6) can be derived in a similar way:

$$v_n[3,3]=\frac{0.8128101475+0.4462504887I\,t+0.0847152093t^2+0.0560717919I\,t^3}{1+0.5089366486I\,t+0.0778056627t^2+0.0526154048I\,t^3}$$

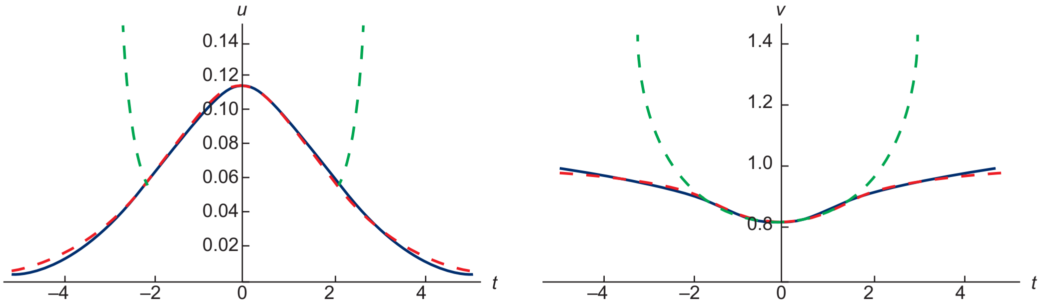

Figure 1. The compared results for the GDTM solutions (green), the GDTM-Padé solutions (blue) and the exact solutions (red) of eq. (1) when n = 4 *(for color image see journal web site)*

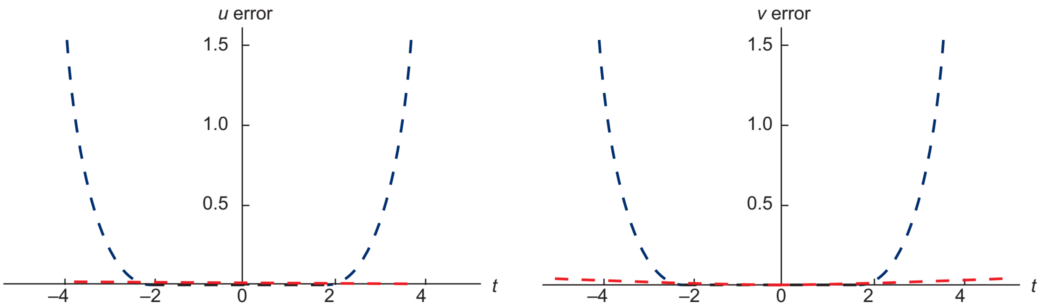

Figure 2. The absolute error curves for the GDTM solutions (blue) and the GDTM-Padé solutions (red) of eq. (1) when *n* =4 *(for color image see journal web site)*

In order to illustrate the efficiency of the GDTM-Padé technique, we plot the GDTM solutions ($|u_{n.6}|$), the GDTM-Padé solutions ($|u_{n.6}[3, 3]|$), and the exact solution ($|u_n|$) of eq. (1) in the left side of fig. 1. The comparisons for the $|v_{n.6}|$, $|v_n[3, 3]|$, and $|v_n|$ are shown in the right side of fig. 1. Figure 2 shows the absolute errors of the modulus of the GDTM solutions and the GDTM-Padé solutions. Obviously, the GDTM-Padé technique performs better than the GDTM method. We remark that the GDTM solutions are in good accordance with the exact solutions in the small interval $-2 \leq t \leq 2$, and high errors appear when $|t| > 2$. By the GDTM-Padé technique, the convergence domains of the approximations are improved largely. In tab. 2, we list the absolute errors of the modulus of GDTM solutions and GDTM-Padé solutions. The numerical results confirm that the GDTM-Padé technique is efficient for solving the Belov-Chaltikian equation.

Conclusions

This paper deals with the Belov-Chaltikian lattice equation by using the GDTM-Padé technique. The numerical results confirm the effectiveness and advantage of this method over the original GDTM method. In the future work, we will further extend this method to other non-linear equations.

Table 2. Comparisons of the absolute errors for the GDTM solutions and the GDTM-Padé solutions to eq. (1) with n = 4

t	$\|\|u_n\| - \|u\|\|$		$\|\|v_n\| - \|v\|\|$	
	GDTM	GDTM-Padé	GDTM	GDTM-Padé
–5	9.20836	$1.57\cdot10^{-3}$	9.88742	$2.14\cdot10^{-2}$
–4	2.25264	$4.06\cdot10^{-3}$	2.33297	$1.25\cdot10^{-2}$
–3	0.32290	$1.58\cdot10^{-3}$	0.32572	$4.86\cdot10^{-3}$
–2	$3.66\cdot10^{-4}$	$2.19\cdot10^{-4}$	$1.77\cdot10^{-2}$	$7.58\cdot10^{-4}$
–1	$2.69\cdot10^{-5}$	$2.34\cdot10^{-6}$	$9.55\cdot10^{-5}$	$9.57\cdot10^{-6}$
0	0	0	0	0
1	$2.69\cdot10^{-5}$	$2.34\cdot10^{-6}$	$9.55\cdot10^{-5}$	$9.57\cdot10^{-6}$
2	$3.66\cdot10^{-4}$	$2.19\cdot10^{-4}$	$1.77\cdot10^{-2}$	$7.58\cdot10^{-4}$
3	0.32290	$1.58\cdot10^{-3}$	0.32527	$4.86\cdot10^{-3}$
4	2.25264	$4.06\cdot10^{-3}$	2.33297	$1.25\cdot10^{-2}$
5	9.20836	$1.57\cdot10^{-3}$	9.88742	$2.14\cdot10^{-2}$

Acknowledgments

The work is supported by the National Natural Science Foundation of China (11201422), the Natural Science Foundation of Zhejiang Province (Y6110639, LQ12A01017), and the project Y201326815 of Education Department of Zhejiang Province.

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Paper submitted: January 18, 2013
Paper revised: April 26, 2013
Paper accepted: May 1, 2013