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**1-D HEAT CONDUCTION IN A FRACTAL MEDIUM
A solution by the Local Fractional Fourier Series Method**

by

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Short paper

DOI: 10.2298/TSC1130303041Z

In this communication 1-D heat conduction in a fractal medium is solved by the local fractional Fourier series method. The solution developed allows relating the basic properties of the fractal medium to the local heat transfer mechanism.

Key words: fractal medium, heat-conduction, local fractional Fourier series method

Introduction

The conventional 1-D heat conduction equation in smooth area of a rod described by the continuum approach is [1]:

$$\frac{\partial u(x, t)}{\partial t} = a \frac{\partial^2 u(x, t)}{\partial x^2} \quad (1)$$

The general form of eq. (1) is the space-time-fractional (non-local) heat conduction equation:

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = a \frac{\partial^\beta u(x, t)}{\partial x^\beta}, \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 2 \quad (2)$$

expressed by Caputo derivatives [2]. Equation (2) describes either the Fourier ($\alpha = \beta = 1$) and non-Fourier materials with memory effects (time fractional term, $\alpha < 1$) and space traps (the space fractional term). The non-local (*non-integer order derivative in space and time*) heat-conduction problems have been solved by numerous approximate analytical methods such as the Adomian decomposition method (ADM) [1], the variational iteration method (VIM) [3-5], the homotopy perturbation method (HPM) [6, 7], and the heat-balance integral method (HBIM) [8, 9].

However, eq. (2) refers to smooth spaces and cannot be applied directly to case where the heat propagates through fractal media. To account the non-smoothness of the fractal heat transfer, the fractional complex transform method [10] allows deriving the *local fractional* directly from eq. (1), namely:

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$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = a \frac{\partial^{2\alpha} u(x, t)}{\partial x^{2\alpha}}, \quad 0 < \alpha \leq 1 \quad (3)$$

with fractal initial conditions:

$$\frac{\partial^\alpha u(x, t)}{\partial x^\alpha} = 0, \quad u(0, t) = \frac{\partial^\alpha u(l, t)}{\partial x^\alpha} = 0, \quad u(x, 0) = \frac{x^\alpha}{\Gamma(1 + \alpha)} \quad (4a, b, c)$$

The local fractional differential operator is defined through [10-14]

$$D_x^{(\alpha)} f(x_0) = f^{(\alpha)}(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha [f(x) - f(x_0)]}{(x - x_0)^\alpha} \quad (5)$$

with $\Delta^\alpha [f(x) - f(x_0)] \cong \Gamma(1 + \alpha) \Delta [f(x) - f(x_0)]$.

In eq. (3) the fractal heat diffusivity of the medium $a = K^{2\alpha} / \rho^\alpha c^\alpha$ is relate to the fractal dimensions of the medium. It is shown that fractal characteristics of the materials determine the heat transfer properties and is applied to describe the fractal heat conduction problems for the slag waste heat recovery. The fractal equation needs methods different from those applied to the non-local ones and local version of the aforementioned approached have been developed, among them: local fractional VIM [11], local fractional Fourier series method (FSM) [12, 14], local fractional Fourier transforms [7, 14], and the local fractional Laplace transforms [7, 14].

This brief note presents an efficient solution to a local fractional heat-conduction equation in a fractal heat medium performed by the local fractional FSM.

Local fractional Fourier series method

If $f(x)$ is $2l$ -periodic function, then the local fractional Fourier series of $f(x)$ is defined as [12, 14]:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_n \cos_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} + b_n \sin_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} \right) \quad (6)$$

The local fractional Fourier coefficients given by [5, 7]:

$$a_k = \frac{1}{l^\alpha} \int_{-1}^1 f(x) (dx)^\alpha, \quad a_k = \frac{1}{l^\alpha} \int_{-1}^1 f(x) \cos_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} (dx)^\alpha, \\ b_k = \frac{1}{l^\alpha} \int_{-1}^1 f(x) \sin_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} (dx)^\alpha \quad (7a, b, c)$$

where the sine and cosine functions in fractal space are [5-7]:

$$\sin_\alpha x^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{\alpha(2k+1)}}{\Gamma[1 + \alpha(2k+1)]}, \\ \cos_\alpha x^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2\alpha k}}{\Gamma(1 + 2\alpha k)}, \quad x \in R \quad \text{and} \quad 0 < \alpha \leq 1 \quad (7d, e)$$

Solution

We suggest a particular solution $u(x, t) = \phi(x)T(t)$ of eq. (3) and taking the local fractional derivative, we can obtain:

$$\frac{T^{(\alpha)}(t)}{\alpha T(t)} = \frac{\phi^{(2\alpha)}(x)}{\phi(x)} \quad (8)$$

If each side of eq. (8) is equal to the constant $-\lambda^{2\alpha}$, then we get:

$$\phi^{(2\alpha)} + \lambda^{2\alpha}\phi = 0, \quad T^{(\alpha)} + a\lambda^{2\alpha}T = 0 \quad (9a,b)$$

In view of eq. (9a, b), we arrive at:

$$\phi = A\cos_\alpha(\lambda^\alpha x^\alpha) + B\sin_\alpha(\lambda^\alpha x^\alpha), \quad T + CE_\alpha(-\lambda^{2\alpha}at^\alpha) \quad (10a,b)$$

where A , B , and C are constants to be determined.

Therefore, the general solution can be expressed as:

$$u(x, t) = CE_\alpha(-l^{2\alpha}at^\alpha)[A\cos_\alpha(l^\alpha x^\alpha) + B\sin_\alpha(l^\alpha x^\alpha)] \quad (11)$$

In view of eqs. (4a) and (11), we have:

$$u(x, t) = ACE_\alpha(-\lambda^{2\alpha}at^\alpha)\cos_\alpha(\lambda^\alpha x^\alpha) \quad (12a)$$

From eqs. (4b) and (11) we have that $\sin_\alpha(l^\alpha x^\alpha) = 0$ so that $\lambda = m\pi/l$. Then, we obtain:

$$u(x, t) = ACE_\alpha\left(\frac{-m^{2\alpha}\pi^{2\alpha}at^\alpha}{l^{2\alpha}}\right)\cos_\alpha\left(\frac{m^\alpha\pi^\alpha x^\alpha}{l^\alpha}\right), \quad m \in \{0\} \cup \{N\} \quad (12b)$$

Considering the fractal condition, eq. (4c), we can construct the local fractional Fourier series as:

$$u(x, t) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m E_\alpha\left(\frac{-m^{2\alpha}\pi^{2\alpha}at^\alpha}{l^{2\alpha}}\right)\cos_\alpha\left(\frac{m^\alpha\pi^\alpha x^\alpha}{l^\alpha}\right) \quad (13a)$$

which leads to:

$$u(x, 0) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos_\alpha\left(\frac{m^\alpha\pi^\alpha x^\alpha}{l^\alpha}\right) \quad (13b)$$

with local fractional Fourier coefficients:

$$a_0 = \frac{1}{l^\alpha} \int_{-l}^l \left[\frac{x^\alpha}{\Gamma(1+\alpha)} \right] (dx)^\alpha = 0 \quad (14)$$

and

$$A_m = \frac{1}{l^\alpha} \int_{-l}^l \frac{x^\alpha}{\Gamma(1+\alpha)} \cos_\alpha\left(\frac{m^\alpha\pi^\alpha x^\alpha}{l^\alpha}\right) (dx)^\alpha = \left(\frac{2m^{2\alpha}\pi^{2\alpha}}{l^{2\alpha}}\right) \cos_\alpha(m^\alpha\pi^\alpha) \quad (15)$$

Hence, the local fractional series solution is:

$$u(x, t) = \sum_{m=1}^{\infty} \left(\frac{2m^{2\alpha}\pi^{2\alpha}}{l^{2\alpha}}\right) \cos_\alpha(m^\alpha\pi^\alpha) E_\alpha\left(\frac{-m^{2\alpha}\pi^{2\alpha}at^\alpha}{l^{2\alpha}}\right) \cos_\alpha\left(\frac{m^\alpha\pi^\alpha x^\alpha}{l^\alpha}\right) \quad (16)$$

Equation (16) is a Weierstrass-like but discontinuous function and satisfies the fractal conditions [3-7]:

$$|u(x, t) - u(x_0, t)| \leq C_1 |x - x_0|^\alpha \quad \text{and} \quad |u(x, t) - u(x, t_0)| \leq C_2 |t - t_0|^\alpha \quad (\text{constants } C_1, C_2) \quad (17a,b)$$

Conclusions

This brief communication refers to 1-D heat-conduction equation in fractal space described by a local fractional equation and a solution by the local fractional Fourier series method. The solution developed is a nowhere-differentiable function related to the local fractal behavior of the medium (material).

Acknowledgments

The work was supported by National scientific and technological support projects (No.2012BAE09B00), the National Natural Science Foundation of China (No. 11126213 and No. 61170317) and the National Natural Science Foundation of Hebei Province (No. E2013209123).

Nomenclature

a	– heat diffusivity ($= k/C_V$), [m^2s^{-1}]	x	– space co-ordinate, [m]
C_V	– heat capacity, [Jkg^{-1}]	<i>Greek symbols</i>	
k	– heat conductivity, [$Wm^{-2}K^{-1}$]	α	– time fractional order, [–]
l	– period of $f(x)$	β	– fractional-space order, [–]
$u(x,t)$	– the temperature function, [K]		
t	– time, [s]		

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