# A NOTE ON THE INTEGRAL APPROACH TO NON-LINEAR HEAT CONDUCTION WITH JEFFREY'S FADING MEMORY

by

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*Integral approach by using approximate profile is successfully applied to heat conduction equation with fading memory expressed by a Jeffrey's kernel. The solution is straightforward and the final form of the approximate temperature profile clearly delineates the "viscous effects" corresponding to the classical Fourier law and the relaxation (fading memory). The optimal exponent of the approximate solution is discussed in case of Dirichlet boundary condition.*

Key words: *non-linear diffusion, fading memory, Jeffrey kernel, integral balance approach, approximate solution*

## Introduction

Diffusion phenomena, of heat or mass, are generally described as a consequence of the conservation law by the relationship [1] $\rho C_p(\partial T/\partial t) = -\partial q/\partial x$. Then, with the assumption that the flux $q(x, t)$ is proportional to the gradient of the temperature, we have:

$$q(x,t) = -k\frac{\partial T(x,t)}{\partial x} \Rightarrow \rho C_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \tag{1a,b}$$

In fact, (1a) is a rate equation defining the transport coefficient (heat conductivity) $k$; then applying (1a) we get the classical Fourier heat conduction eq. (1b). However, this equation defines an infinite speed of propagation of the flux which is unphysical because the real processes have finite speeds and there is a time-delay in the propagation of the diffusant into medium when a disturbance at the interface takes place. Therefore, a damping function relating the model to the real processes has to correct the unphysical eq. (1b).

In heat conduction the damping function represented by a Volterra type integral to model a finite sped of heat diffusion in rigid conductors was conceived by Cattaneo [2]. This approach constitutes a generalized Fourier's law with a linear superposition of the heat flux and its time derivative related to its history [3]. With this approach, the heat flux and the relaxation (damping) function $F(x, t)$ are assumed to satisfy the constitutive equation [4]:

$$q(x,t) = -\int_0^\infty F(x,t)\nabla t(x,t-\tau)\,\mathrm{d}\tau \tag{2}$$

Hence, the heat flux depends on the time variable not only *via* the present time $\tau$, but also *via* its past history related to the time-delay $\tau$. Considering the rigid heat conductors as homogeneous materials in general, the damping function does not depend on the space coordinate,

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*i. e.* $F(x, t) \to F(t)$. In the classical Fourier (Fick) theory the heat (mass) flux is related to the temperature (concentration) gradient linearly $q = -k\partial T/\partial x$, where $k$ a positive constant is termed conductivity (diffusivity). When $\partial T/\partial x$ is time-independent, then eq. (I) and (2) reduces to the Fourier (Fick) law [4] with $k = \int_0^\infty F(x,t)\,\mathrm{d}\tau$. The Cattaneo equation can be expressed as an integral over the history of the gradient as:

$$q = -\frac{k}{\tau}\int_{-\infty}^{t} \mathrm{e}^{-\frac{t-s}{\tau}} \frac{\partial T(x,s)}{\partial x}\mathrm{d}s \tag{3}$$

Here the function $F(t) = \exp[-(t-s)/\tau]$ is a relaxation kernel of Jeffrey's type [1, 5]. Then, the diffusion equation becomes [1]:

$$\frac{\partial T(x,t)}{\partial t} = -\frac{k}{\tau}\int_{0}^{t} \mathrm{e}^{-\frac{t-s}{\tau}} \frac{\partial^2 T(x,t)}{\partial x^2}\mathrm{d}s \tag{4}$$

From the energy conservation equation and time derivatives from the heat flux, and the internal energy [4] we have, accordingly:

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,s)}{\partial x^2} + \frac{a_2}{\tau}\int_{0}^{t} e^{\frac{t-s}{\tau}} \frac{\partial^2 T(x,s)}{\partial x^2}\mathrm{d}s \tag{5a}$$

$$a_1 = \frac{k_1}{\rho C_p}, \quad a_2 = \frac{k_2}{\rho C_p} \tag{5b}$$

## Integral-balance solution

Now, we have the heat conduction eq. (5a) with exponential Jeffrey's kernel and an initial condition:

$$T(x,0) = 0, \quad 0 \le x \le \infty \tag{6a}$$

and boundary conditions:

$$T(0,t) = T_0, \quad T(x,t) = 0, \quad x \to \infty \tag{6b,c}$$

The heat-balance integral method suggests a finite speed and penetration depth $\delta(t)$ of the diffusant into the medium. Hence, applying integration with respect to the space co-ordinate $x$ from 0 to $\delta(t)$ [6, 7] to (5a) we have:

$$\int_0^{\delta_J} \frac{\partial T(x,t)}{\partial t}\mathrm{d}x = \int_0^{\delta_j} a_1 \frac{\partial^2 T(x,t)}{\partial x^2}\mathrm{d}x + \int_0^{\delta_J}\left[\frac{a_2}{\tau}\int_0^t \mathrm{e}^{\frac{t-s}{\tau}} \frac{\partial^2 T(x,s)}{\partial x^2}\mathrm{d}s\right]\mathrm{d}x \tag{7a}$$

Then, applying the Leibniz rule to the left-side of (7a) we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\delta_J} T(x,t)\,\mathrm{d}x = \int_0^{\delta_J} a_1 \frac{\partial^2 T(x,t)}{\partial^2 x}\mathrm{d}x + \int_0^{\delta_J}\left[\frac{a_2}{\tau}\int_0^t \mathrm{e}^{\frac{t-s}{\tau}} \frac{\partial^2 T(x,s)}{\partial x^2}\mathrm{d}s\right]\mathrm{d}x \tag{7b}$$

Further, replacing the condition (6c) [6, 7] by $T(\delta, t) = 0$ and $\partial T(\delta, t)/\partial x = 0$, as well as the function $T(x, t)$ by the approximate parabolic profile $T_a(x, t) = T_0(1 - x/\delta)^n$ [8], in eq. (7b) we obtain an ordinary differential equation about $\delta(t)$, namely:

$$\frac{1}{2}\frac{\mathrm{d}\delta_J^2}{\mathrm{d}t} = a_1 \frac{n(n+1)}{\delta_J} + \frac{a_2}{\tau} n(n+1)\left[\tau\left(1 - \mathrm{e}^{-\frac{t}{\tau}}\right)\right] \tag{8}$$

$$\delta_J^2 = 2n(n+1)\left(a_1 t + a_2 \tau \mathrm{e}^{-\frac{t}{\tau}}\right) \Rightarrow \delta_J = \sqrt{2n(n+1)}\sqrt{a_1 t}\sqrt{\left(1+\frac{a_2}{a_1}\frac{\tau}{t}\mathrm{e}^{-\frac{t}{\tau}}\right)} \tag{9a,b}$$

With $k_2 = 0$ and $\tau/t \to 0$ (large times) we have the classical diffusion equation and the classical long-time result [6, 7] $\delta_0 = (a_1 t)^{1/2}[2n(n+1)]^{1/2}$. The introduction of the concept of the final penetration depth $\delta(t)$ is, in fact, *ad hoc*, definition of a damping function in the approximate profile. Thus, from mathematical point of view, the inexact approximate profile satisfies the diffusion equation in average, *i. e.* satisfying the energy balance, but on other hand, this is a profile with a relaxation function.

Because the approximate profile $T_a = T_0(1 - x/\delta)^n$ satisfies the Goodman's conditions $T(\delta, t) = 0$ and $\partial T(\delta, t)/\partial x = 0$ with any value of the exponent $n$ [8] the accuracy of approximation depends on the choice of the exponent $n$ and refers to consequent optimizations of the parabolic profile [8-11] . Therefore, the dimensionless approximate profile is:

$$\frac{T_a(x,t)}{T_0} = \left[1 - \frac{x}{(\sqrt{a_1 t}\sqrt{2n(x+1)})F_a(t)}\right]^n = \left(1 - \frac{x}{\sqrt{a_1 t}F_n F_a(t)}\right)^n \tag{10}$$

The approximate fading term $F_a(t) = [1+(a_2/a_1)(\tau/t)\exp(-t/\tau)]^{1/2}$ represents the ratio $\delta_J/\delta_0$ and shows the short time elastic effect of the memory function, which fades in time.

The optimal exponent of the parabolic profile when $k_2 = 0$ (Fourier law) is discussed in the literature [9-11] and the problem at issue refers to adequate function $F_n = [2n(n+1)]^{1/2}$ in eq. (10). The common approach is to minimize the $L_2$ norm (expressed as function of $n$ and $t$). We will skip the cumbersome calculations and use the fact that $F_a(t)$ decays rapidly in time. Then, we may suggest that the optimal exponent, taking into account that in accordance with the integral approach *the approximate profile satisfies in average* the governing eq. (5b), can be defined from the reduced model with $k_2 = 0$. With this approach we have several cases (Dirichlet boundary condition) of optimal exponents, namely (a) $n_0^T \approx 1.75$, when the approximate profile is calibrated at $x = 0$ [8, 10, 11]. This approach avoids the condition (11); (b) $n_M^T = 2.235$, obtained after minimization of the $L_2$ norm [9]; (c) $n_\eta^T = 1.5047$, when the minimization of the $L_2$ norm is performed through a preliminary transform of the governing equation by the similarity variable $\eta = x/2(a_1 t)^{1/2}$ [10] . Hence, a parabolic profile with either $n_M^T = 2.235$ or $n_\eta^T = 1.5047$ is applicable, depending on the required actuary of approximation (see details in ref. [10]).

The form of the time-dependent penetration depth $\delta_J(t)$ allows estimating the initial thermal penetration speed $\Delta_\delta = \mathrm{d}\delta_J/\mathrm{d}t$ at $t = 0$. In this context, we have to take into account that for the Fourier conduction equation the penetration depth $\delta_0 = (a_1 t)^{1/2}[2n(n+1)]^{1/2}$ introduces a singularity because $\mathrm{d}\delta_0/\mathrm{d}t$ is infinite at $t = 0$. This problem has been studied by Tzou and Chiu [12] for the thermal penetration estimated by the heat-balance integral (a cubic profile with $n = 3$) when the heat conduction is modelled by a dual-phase lag equation (hyperbolic) with two relaxation times. Now, with the estimate (9b) we have:

$$\frac{\mathrm{d}\delta_J}{\mathrm{d}t} = \frac{1}{\sqrt{2}}\sqrt{n(n+1)}\frac{a_1 - a_2\mathrm{e}^{-t/\tau}}{\sqrt{a_1 t + a_2\tau\mathrm{e}^{-t/\tau}}} \Rightarrow \Delta_\delta = \frac{1}{\sqrt{2}}\sqrt{n(n+1)}\frac{a_1 - a_2}{\sqrt{a_2\tau}} \tag{11a,b}$$

Hence, with $n = 2$ for example, we have $\Delta_{\delta(n=2)} = 3^{1/2}[(a_1 - a_2)/(a_2\tau)^{1/2}$, while with $n = 3$ the speed is $\Delta_{\delta(n=2)} = 6^{1/2}/(a_2\tau)^{1/2}$.

With these estimates, we obtain an adequate physically sound approximate solution of the heat conduction equation with a fading memory, precisely:

(1) Qualitatively, we have a model with a relaxation that assumes a finite speed of the heat penetration into the medium.
(2) The heat balance integral approach results in penetration depth depending on the both the time and the relaxation parameter (Deborah number), and has a finite speed , thus avoiding the problem with the singularity when the Fourier equation is only considered.

## Conclusions

The integral approach based on the assumption of a final penetration depth was successfully applied to heat conduction with a relation function of Jeffrey's type. The final form of the approximate profile clearly discriminates the short time relation effect and the large time approximation established to the Fourier heat conduction model. The form of the relaxation function in the approximate profile indicates that its time-evolution depends on dimensionless variables: the similarity variable $\eta = x/2(a_1 t)^{1/2}$ and the ratio $D_0 = \tau/t$, (analogue of the Deborah number in the viscoelasticity [13]).

The introduction of the memory term in the heat conduction equation results in a relationship about the thermal penetration depth which allows estimating analytically its speed at $t = 0$. This speed is finite, in contrast to the case of the Fourier equation where the penetration speed has a singularity at $t = 0$.

## Nomenclature

$a_1$ – effective heat diffusivity (Fourier law), $[\mathrm{m^2 s}]$
$a_2$ – elastic heat diffusivity (Cattaneo equation), $[\mathrm{Wm^{-1}K^{-1}}]$
$C_p$ – heat capacity, $[\mathrm{Jkg^{-1}}]$
$k$ – heat conductivity, $[\mathrm{Wm^{-1}K^{-1}}]$
$k_1$ – effective heat conductivity (Fourier law), $[\mathrm{Wm^{-1}K^{-1}}]$
$k_2$ – elastic heat conductivity (Cattaneo equation),$[\mathrm{Wm^{-1}K^{-1}}]$
$n$ – exponent of the approximate profile, [–]
$n_M^T$ – optimal exponent of Myers [9]
$n_\eta^T$ – optimal exponent of Hristov [10] (= 1.0547)
$n_0^T$ – exponent defined for $k_2 = 0$ at $x = 0$, [–]
$T$ – temperature, [K]
$q$ – heat flux, $[\mathrm{Wm^{-2}}]$
$s$ – dummy variable, [–]
$t$ – time, [s]
$x$ – space co-ordinate, [m]

*Greek letters*

$\delta$ – penetration depth, [m]
$\delta_0$ – penetration depth with $k_2 = 0$ (Fourier law), [m]
$\delta_J$ – penetration depth accounting for the Jeffrey's relation kernel, [m]
$\rho$ – density, $[\mathrm{kgm^{-3}}]$
$\tau$ – time delay (relaxation time), [s]

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