THE YANG-FOURIER TRANSFORMS TO HEAT-CONDUCTION IN A SEMI-INFINITE FRACTAL BAR

by

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1-D fractal heat-conduction problem in a fractal semi-infinite bar has been developed by local fractional calculus employing the analytical Yang-Fourier transforms method. The simplicity and the accuracy of the method are discussed.

Key words: *heat-conduction equation, fractal bar, Yang-Fourier transforms, local fractional calculus*

Introduction

The number of applicable mathematical and engineering problems successfully solved by the tools of the fractional calculus is continuously growing in last five decades [1-7]. Most of the fractional differential equations have exact analytic solutions, whilst others need either analytical approximations or numerical techniques to be applied, among them: fractional Fourier and Laplace transforms [8], heat-balance integral method [9-11], variational iteration method (VIM) [12-14], decomposition method [15], homotopy perturbation method [16], fractional variational homotopy perturbation iteration method [17], finite element method [18], fractional sub-equation method [19], Mellin integral transform [20], homotopy analysis method [21, 22], finite difference method [23], Taylor series expansion method [24], wavelet operational method [25], *etc*.

The memory properties of fractional derivatives and integrals of the classical calculus implicitly mean smooth spaces and fails when local and fractal behaviors should be modeled. However, problems in fractal media can be successfully solved by local fractional calculus theory with problems for non-differential functions [25-32]. Local fractional differential equations have been applied to model complex systems of fractal physical phenomena [30-40] with a variety of methods such as local fractional VIM [35-37], local fractional Fourier series method [38], Yang-Fourier transform [39, 40], Yang-Laplace transform [40], *etc*. The heat conduction in fractal media requires modelling by local fractional derivatives as it was demonstrated a series of articles [32, 33, 35], and the reference therein).

The present communication addresses a transient heat conduction problem in a fractal semi-infinite bar solved by the Yang-Fourier transform [39, 40].

Yang-Fourier transform and its properties

Suppose that $f(x)$ is local fractional continuous at the interval $(-\infty, \infty)$, we denote as $f(x) \in C_\alpha(-\infty, \infty)$ (see [32, 33, 35], and the reference therein).

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Let $f(x) \in C_\alpha(-\infty, \infty)$. The Yang-Fourier transform is written in the form [30, 31, 39, 40]:

$$F_\alpha\{f(x)\} = f_\omega^{F,\alpha}(\omega) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} E_\alpha(-i^\alpha \omega^\alpha x^\alpha) f(x)(\mathrm{d}x)^\alpha \tag{1}$$

Then, the local fractional integration is given by [30-32, 35-37]:

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(\mathrm{d}x)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha \tag{2}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, ...\}$ and $[t_j, t_{j+1}\}$, $j = 0, ..., N-1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a, b]$.

If $F_\alpha\{f(x)\} \equiv f_\omega^{F,\alpha}(\omega)$, then its inversion formula takes the form [30, 31, 39, 40]:

$$f(x) = F_\alpha^{-1}[f_\omega^{F,\alpha}(\omega)] = \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} E_\alpha(i^\alpha \omega^\alpha x^\alpha) f_\omega^{F,\alpha}(\omega)(\mathrm{d}\omega)^\alpha \tag{3}$$

Some properties are shown as it follows [30, 31]:

Let $F_\alpha\{f(x)\} \equiv f_\omega^{F,\alpha}(\omega)$, and $F_\alpha\{g(x)\} = f_\omega^{F,\alpha}(\omega)$, and let be two constants. Then we have:

$$F_\alpha\{af(x) + bg(x)\} = aF_\alpha\{f(x)\} + bF_\alpha\{g(x)\} \tag{4}$$

If $\lim_{|x| \to \infty} f(x) = 0$, then we have:

$$F_\alpha\{f^{(\alpha)}(x)\} = i^\alpha \omega^\alpha F_\alpha\{f(x)\} \tag{5}$$

In eq. (5) the local fractional derivative is defined as:

$$f^{(\alpha)}(x_0) = \left.\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha[f(x) - f(x_0)]}{(x - x_0)^\alpha} \tag{6}$$

where $\Delta^\alpha[f(x) - f(x_0)] \cong \Gamma(1+\alpha)\Delta[f(x) - f(x_0)]$.

As a direct result, repeating this process, when:

$$f(0) = f^{(\alpha)}(0) = \ldots = f^{[(k-1)\alpha]}(0) = 0 \tag{7}$$

we get

$$F_\alpha\{f^{(k\alpha)}(x)\} = i^{k\alpha} \omega^{k\alpha} F_\alpha\{f(x)\} \tag{8}$$

Heat conduction in a fractal semi-infinite bar

When a fractal body is subjected to a boundary perturbation, then the heat diffuses in depth modeled by a constitutive relation where the rate of fractal heat flux $\vec{q}(x, y, z, t)$ is proportional to the local fractional gradient of the temperature [32], namely:

$$\vec{q}(x, y, z, t) = -K^{2\alpha} \nabla^\alpha T(x, y, z, t) \tag{9}$$

Here the pre-factor $K^{2\alpha}$ is the thermal conductivity of the fractal material. Therefore, the fractal heat conduction equation without heat generation was suggested in [32] as:

$$K^{2\alpha} \frac{\mathrm{d}^{2\alpha} T(x, y, z, t)}{\mathrm{d}x^{2\alpha}} - \rho_\alpha c_\alpha \frac{\mathrm{d}^\alpha T(x, y, z, t)}{\mathrm{d}x^\alpha} = 0 \tag{10}$$

where ρ_α and c_α are the density and the specific heat of material, respectively.

The fractal heat-conduction equation with a volumetric heat generation $g(x, y, z, t)$ can be described as [32]:

$$K^{2\alpha}\nabla^{2\alpha}T(x,y,z,t)+g(x,y,z,t)-\rho_\alpha c_\alpha\frac{\partial^\alpha T(x,y,z,t)}{\partial t^\alpha} \tag{11}$$

The 1-D fractal heat-conduction equation [32] reads as:

$$K^{2\alpha}\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}}-\rho_\alpha c_\alpha\frac{\partial^\alpha T(x,t)}{\partial t^\alpha}=0,\quad 0<x<\infty,\quad t>0 \tag{12a}$$

with initial and boundary conditions are:

$$\frac{\partial^\alpha T(0,t)}{\partial x^\alpha}=E_\alpha(t^{\alpha)},\ T(0,t)=0 \tag{12b, c}$$

The dimensionless forms of (12a, b, c) are [35]:

$$\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}}=\frac{\partial^\alpha T(x,t)}{\partial x^\alpha}=0 \tag{13a}$$

$$\frac{\partial^\alpha T(0,t)}{\partial x^\alpha}=E_\alpha(t^\alpha),\ T(0,t)=0 \tag{13b}$$

Based on eq. (12a), the local fractional model for 1-D fractal heat-conduction in a fractal semi-infinite bar with a source term $g(x, t)$ is:

$$K^{2\alpha}\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}}-\rho_\alpha c_\alpha\frac{\partial^\alpha T(x,t)}{\partial t^\alpha}=g(x,t),\ -\infty<x<\infty,\ t>0 \tag{14a}$$

with

$$T(x,0)=f(x),\ -\infty<x<\infty \tag{14b}$$

The dimensionless form of the model (14a, b) is:

$$\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}}-\frac{\partial^\alpha T(x,t)}{\partial t^\alpha}=0,\ -\infty<x<\infty,\ t>0 \tag{15a}$$

$$T(x,0)=f(x),\ -\infty<x<\infty \tag{15b}$$

Solution by the Yang-Fourier transform method

Let us consider that $\mathrm{F}_\alpha\{T(x,t)\}=T_\omega^{F,\alpha}(\omega,t)$ is the Yang-Fourier transform of $T(x, t)$, regarded as a non-differentiable function of x. Applying the Yang-Fourier transform to the first term of eq. (15a), we obtain:

$$F_\alpha\left\{\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}}\right\}=i^{2\alpha}\omega^{2\alpha}T_\omega^{F,\alpha}(\omega,t)=-\omega^{2\alpha}T_\omega^{F,\alpha}(\omega,t) \tag{16a}$$

On the other hand, by changing the order of the local fractional differentiation and integration in the second term of eq.(15a), we get:

$$F_\alpha\left\{\frac{\partial^{2\alpha}T(x,t)}{\partial t^{2\alpha}}\right\}=\frac{\partial^{2\alpha}T_\omega^{F,\alpha}(\omega,t)}{\partial t^\alpha} \tag{16b}$$

For the initial value condition, the Yang-Fourier transform provides:

$$F_\alpha\{T(x,0)\}=T_\omega^{F,\alpha}(\omega,0)=F_\alpha\{f(x)\}=f_\omega^{F,\alpha}(\omega) \tag{16c}$$

Hence, from eqs. (16a, b,c), we obtain:

$$\frac{\partial^{2\alpha} T_{\omega}^{F,\alpha}(\omega,t)}{\partial t^{\alpha}} + \omega^{2\alpha} T_{\omega}^{F,\alpha}(\omega,t) = 0,\ \ t>0,\ \ T_{\omega}^{F,\alpha}(\omega,0) = f_{\omega}^{F,\alpha}(\omega) \tag{17}$$

This is an initial value problem of a local fractional ordinary differential equation with t as independent variable and ω as a parameter. Now, by the help of (A.4), in the Appendix the solution of eq.(17) is:

$$T(\omega,t) = f_{\omega}^{F,\alpha}(\omega) E_{\alpha}(-\omega^{2\alpha} t^{\alpha}) \tag{18a}$$

Consequently, using inversion formula, see eq. (3), we obtain:

$$T(x,t) = \frac{1}{(2\pi)^{\alpha}} \int_{-\infty}^{\infty} E_{\alpha}(i^{\alpha}\omega^{\alpha}x^{\alpha}) f_{\omega}^{F,\alpha}(\omega) E_{\alpha}(-\omega^{2\alpha}t^{\alpha})(\mathrm{d}\omega)^{\alpha} = (Mf)(x) \tag{18b}$$

$$M_{\omega}^{F,\alpha} = \frac{1}{(2\pi)^{\alpha}} E_{\alpha}(-\omega^{2\alpha}t^{\alpha}) \tag{18c}$$

In view of (A.9), we get:

$$F_{\alpha}\left\{E_{\alpha}\left(-\frac{\omega^{2\alpha}}{C^{2\alpha}}\right)\right\} = \frac{C^{\alpha}\pi^{\frac{\alpha}{2}}}{\Gamma(1+\alpha)} E_{\alpha}\left(-\frac{C^{2\alpha}\omega^{2\alpha}}{4^{\alpha}}\right) \tag{19a}$$

Let $C^{2\alpha}/4\alpha = t^{\alpha}$. Then we obtain:

$$F_{\alpha}\left\{E_{\alpha}\left(-\frac{\omega^{2\alpha}}{4^{\alpha}t^{\alpha}}\right)\right\} = \frac{4^{\alpha}t^{\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}}}{\Gamma(1+\alpha)} + E_{\alpha}(-\omega^{2\alpha}t^{\alpha}) = \frac{4^{\alpha}t^{\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}}(2\pi)^{\alpha}}{\Gamma(1+\alpha)} M_{\omega}^{F,\alpha}(\omega) \tag{19b}$$

Therefore, $M_{\omega}^{F,\alpha}(\omega)$ have the inverse, namely:

$$\frac{1}{(2\pi)^{\alpha}} \int_{-\infty}^{\infty} E_{\alpha}(i^{\alpha}\omega^{\alpha}x^{\alpha}) M_{\omega}^{F,\alpha}(\omega)(\mathrm{d}\omega)^{\alpha} = \frac{\Gamma(1+\alpha)}{4^{\alpha}t^{\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}}(2\pi)^{\alpha}} E_{\alpha}\left(-\frac{\omega^{2\alpha}}{4^{\alpha}t^{\alpha}}\right) \tag{19c}$$

Finally, we obtain:

$$T(x,t) = (Mf)(x) = \frac{\Gamma(1+\alpha)}{4^{\alpha}t^{\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}}} \int_{-\infty}^{\infty} f(\varsigma) E_{\alpha}\left(-\frac{(x-\varsigma)^{2\alpha}}{4^{\alpha}t^{\alpha}}\right)(\mathrm{d}\varsigma)^{\alpha} \tag{20}$$

Conclusions

The communication, successfully presented an analytical solution of 1-D heat conduction in fractal semi-infinite bar through the Yang-Fourier transform of non-differentiable functions. The solution clearly shows show the accuracy and reliable results. The method is applied to a fractional partial equation defined on a Cantor set in a manner useful for practical purposes.

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Nomenclature

F_{α} – Yang-Fourier transform of $f(x)$
k – heat conductivity, [$\mathrm{Wm^{-2}K^{-1}}$]
t – time, [s]
u – the temperature function, (= x, t), [K]
x – space co-ordinate, [m]
α – time fractional dimensional order, [–]

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Appendix

In a Cantor type circle co-ordinate [32], we have:

$$\begin{cases} x^{\alpha} = R^{\alpha} \cos_{\alpha} \theta^{\alpha} \\ y^{\alpha} = R^{\alpha} \sin_{\alpha} \theta^{\alpha} \end{cases} (0 < R < \infty, \quad 0 \le \theta \le 2\pi) \tag{A.1}$$

we have [32]:

$$\int\int_{R} F(x, y) \mathrm{d}S^{(\beta)} = \int\int_{S^{(\beta)}} F[x(u, v), y(u, v)] \frac{R^{\alpha}}{\Gamma(1+\alpha)} (\mathrm{d}\theta)^{\alpha} (\mathrm{d}R)^{\alpha} \tag{A.2}$$

For a given fractal region $S = \{(x, y) : x^{2\alpha} + y^{2\alpha} = R^{2\alpha}\}$, its area is [32]:

$$\begin{aligned} \frac{1}{\Gamma(1+\alpha)} \int\int_{S} \mathrm{d}S^{(\beta)} &= \frac{1}{\Gamma^2(1+\alpha)} \int\int_{S} \frac{R^{\alpha}}{\Gamma(1+\alpha)} (\mathrm{d}\theta)^{\alpha} (\mathrm{d}R)^{\alpha} = \\ &= \frac{1}{\Gamma^2(1+\alpha)} \int_0^{2\pi} (\mathrm{d}\theta)^{\alpha} \int_0^{R} \frac{R^{\alpha}}{\Gamma(1+\alpha)} (\mathrm{d}R)^{\alpha} = \frac{(2\pi)^{\alpha}}{\Gamma(1+\alpha)} + \frac{R^{2\alpha}}{\Gamma(1+2\alpha)} \end{aligned} \tag{A.3}$$

For a given fractal region $S^{(\beta)} = \{(x, y) : x^{2\alpha} + y^{2\alpha} = R^{2\alpha}\}$, we have [32]:

$$\begin{aligned} \left[\frac{1}{\Gamma(1+\alpha)} \int_{-4}^{4} E_{\alpha}(-x^{2\alpha})(\mathrm{d}x)^{\alpha}\right]^2 &= \frac{1}{\Gamma^2(1+\alpha)} \int\int_{S^{(\beta)}} E_{\alpha}[-(x^{2\alpha} + y^{2\alpha})](\mathrm{d}x)^{\alpha} (\mathrm{d}y)^{\alpha} = \\ &= \left[\frac{1}{\Gamma(1+\alpha)} \int_0^{2\pi} (\mathrm{d}\theta)^{\alpha}\right] \left[\frac{1}{\Gamma(1+\alpha)} \int_0^{4} E_{\alpha}(-R^{2\alpha}) \frac{R^{\alpha}}{\Gamma(1+\alpha)} (\mathrm{d}R)^{\alpha}\right] = \frac{\pi^{\alpha}}{\Gamma^2(1+\alpha)} \end{aligned} \tag{A.4}$$

Hence, we get:

$$\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}E_{\alpha}(-x^{2\alpha})(\mathrm{d}x)^{\alpha}=\frac{\pi^{\frac{\alpha}{2}}}{\Gamma(1+\alpha)} \tag{A.5}$$

and

$$\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left\{\frac{\Gamma(1+\alpha)}{C^{\alpha}\pi^{\frac{\alpha}{2}}}E_{\alpha}\left[-\frac{x^{2\alpha}}{C^{2\alpha}}\right]\right\}(\mathrm{d}x)^{\alpha}=1 \tag{A.6}$$

where $C > 0$.

If $k > 0$, then local fractional equation [30-32]:

$$\frac{d^{\alpha}y}{\mathrm{d}x^{\alpha}}+ky=0, \quad y(0)=y_0 \tag{A.7}$$

has the one-parameter family of solutions:

$$y(x)=y_0E^{\alpha}(-kx^{\alpha}) \tag{A.8}$$

$$F_{\alpha}\left\{E_{\alpha}\left(-\frac{x^{2\alpha}}{C^{2\alpha}}\right)\right\}=\frac{C^{\alpha}\pi^{\frac{\alpha}{2}}}{\Gamma(1+\alpha)}E_{\alpha}\left(-\frac{C^{2\alpha}\omega^{2\alpha}}{4^{\alpha}}\right) \tag{A.9}$$

Proof

$$F_{\alpha}\left\{E_{\alpha}\left(\frac{x^{2\alpha}}{C^{2\alpha}}\right)\right\}=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}E_{\alpha}\left[\frac{x^{2\alpha}}{C^{2\alpha}}\right]E_{\alpha}(-i^{\alpha}\omega^{\alpha}x^{\alpha})(\mathrm{d}x)^{\alpha} \tag{A.10}$$

In view of (A.9), we rewrite as:

$$\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}E_{\alpha}\left[-\frac{x^{2\alpha}}{C^{2\alpha}}-i^{\alpha}\omega^{\alpha}x^{\alpha})\right](\mathrm{d}x)^{\alpha}=\frac{E_{\alpha}\left(-\frac{C^{2\alpha}\omega^{2\alpha}}{4^{\alpha}}\right)}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}E_{\alpha}\left[\frac{\left(ix-\frac{C^{2}\omega}{2}\right)^{2\alpha}}{C^{2\alpha}}\right](\mathrm{d}x)^{\alpha} \tag{A.11}$$

Hence, we get:

$$\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}E_{\alpha}\left[-\frac{x^{2\alpha}}{C^{2\alpha}}-i^{\alpha}\omega^{\alpha}x^{\alpha}\right](\mathrm{d}x)^{\alpha}=\frac{C^{\alpha}\pi^{\frac{\alpha}{2}}}{\Gamma(1+\alpha)}E_{\alpha}\left(-\frac{C^{2\alpha}\omega^{2\alpha}}{4^{\alpha}}\right) \tag{A.12}$$

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