

A CERTAIN ANALYTICAL METHOD USED FOR SOLVING THE STEFAN PROBLEM

by

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The paper presents an analytic method applied for finding the approximate solution of Stefan problem reduced to the one-phase solidification problem of a plate with the a priori unknown, and varying in time, boundary of the region in which the solution is sought. Proposed method is based on the known formalism of initial extension of a sought function describing the temperature field into the power series, some coefficients of which can be determined with the aid of boundary conditions, and also based on the approximation of a function defining the freezing front location with the broken line, parameters of which can be obtained by solving the appropriate differential equations. Results received by applying the proposed procedure are compared with the results obtained using a classical numerical method for solving the Stefan problem.

Key words: *Stefan problem, solidification, analytic method, boundary element method*

Introduction

The paper presents a new analytical method for finding a solution of the one-phase Stefan problem. The method is interesting especially from the engineer's point of view because of its simplicity and naturalness. It can be used for solving a selected class of problems which can be expressed by means of a one-phase solidification problem of a plate with the *a priori* unknown, varying in time, boundary of a region in which the solution is sought. The method bases on a known formalism of initial extension of the sought function describing the temperature field into the power series, coefficients of which can be determined by using the boundary conditions. The process considered is described as the Stefan problem. Stefan problem is a name of mathematical description of thermal processes in which the phase transition takes place. Examples of processes of that kind can be solidification of pure metals, melting of ice, freezing of water, deep freezing of food articles and others [1, 2]. Solution of the Stefan problem consists in determination of the temperature distribution in considered region and in finding the freezing front location under the assumption of knowing the initial and boundary conditions, as well as the thermo-physical properties of examined environment.

Since determination of the exact solution of Stefan problem is possible only in some 1-D cases [3-5], in most considered cases finding the solution of Stefan problem reduces to obtaining its approximation, less or more close to the exact one. Therefore there are developed various methods of approximate solution of the Stefan problem [6-10]. Except classical methods,

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based for example on the finite difference method, finite element method or boundary element method, also the iteration methods, not requiring to discretize the examined region, appeared to be effective, like the Adomian decomposition method or variational iteration method. In papers [11, 12] the authors have used the Adomian method combined with the appropriate minimization procedure for determining the approximate solution of one-phase Stefan problem, whereas the application of variational iteration method in approximate solution of direct and inverse Stefan problem is discussed in paper [13]. Slightly different approach to the Stefan problem is showed in works [14, 15] where the investigated problem is first reduced to the problem defined in a unit square and next such transformed task is solved with the aid of variational iteration method. The same procedure is used in paper [16] for solution of 1-D moving boundary problem with periodic boundary conditions. Another examples of application of the variational iteration method can be found in papers [17, 18] as well as in references included in these works. Furthermore, Das and Rajeev in paper [19] have introduced an approximate analytic solution of 1-D Stefan problem. Next, in paper [20] there is presented the Stefan problem governed by fractional diffusion equation. Finally, for solving equations of similar kind the heat-balance integral method can be proposed as well [21–23].

Stefan problem

Mathematical model of the one-phase Stefan problem is expressed by means of the following system of equations:

- heat conduction equation describing the field of varying in time and space temperature T in the formed solid phase

$$\frac{\partial T(x, t)}{\partial t} = a \frac{\partial^2 T(x, t)}{\partial x^2}, \quad x \in [\varphi(t), \bar{x}], \quad t \in (0, t^\infty) \quad (1)$$

where x denotes the spatial variable, t – the time, \bar{x} – the half of the plate thickness, t^∞ – the duration of the process, and $\varphi(t)$ is a function describing varying in time location of the freezing front, it means

$$\varphi(t) = \bar{x} - \xi(t) \quad (2)$$

where $\xi(t)$ denotes a function characterizing varying in time thickness of solidified layer and

$$\xi(0) = 0 \quad (3)$$

- condition of temperature continuity on the interface

$$T[\varphi(t), t] = T^*, \quad t \in (0, t^\infty) \quad (4)$$

- condition of energy balance (Stefan condition) on the interface

$$-\lambda \frac{\partial T[\varphi(t), t]}{\partial x} = \gamma \kappa \frac{d\varphi(t)}{dt}, \quad t \in (0, t^\infty) \quad (5)$$

where λ denotes the thermal conductivity, κ describes the latent heat of fusion, and γ is the mass density;

- boundary condition of the first kind on the heat transfer surface

$$T(\bar{x}, t) = \psi(t), \quad t \in (0, t^\infty) \quad (6)$$

where $\psi(t)$ denotes varying in time temperature of the plate surface.

Novel analytic method

Method of solving the above formulated problem is based on the appropriate representation of the function describing the predicted solution in the form of power series. In this case it is the following series:

$$T(x, t) = \sum_{i=0}^{\infty} A_i(t) \frac{[x - \bar{x} + \xi(t)]^i}{i!} \tag{7}$$

where $A_i(t)$ denote the unknown functional coefficients depending on time. These coefficients are determined by using eq. (1) together with conditions (4) and (5).

Relation (7) implies that:

$$\frac{\partial T(x, t)}{\partial x} = \sum_{i=0}^{\infty} A_{i+1}(t) \frac{[x - \bar{x} + \xi(t)]^i}{i!} \tag{8}$$

$$\frac{\partial^2 T(x, t)}{\partial x^2} = \sum_{i=0}^{\infty} A_{i+2}(t) \frac{[x - \bar{x} + \xi(t)]^i}{i!} \tag{9}$$

and

$$\frac{\partial T(x, t)}{\partial t} = \sum_{i=0}^{\infty} \left[A_i'(t) \frac{[x - \bar{x} + \xi(t)]^i}{i!} + \xi'(t) A_{i+1}(t) \frac{[x - \bar{x} + \xi(t)]^i}{i!} \right] \tag{10}$$

Substituting the respectively obtained formulas to eq. (1) we receive:

$$\begin{aligned} \sum_{i=0}^{\infty} \left[A_i'(t) \frac{[x - \bar{x} + \xi(t)]^i}{i!} + \xi'(t) A_{i+1}(t) \frac{[x - \bar{x} + \xi(t)]^i}{i!} \right] &= \\ &= a \sum_{i=0}^{\infty} A_{i+2}(t) \frac{[x - \bar{x} + \xi(t)]^i}{i!} \end{aligned} \tag{11}$$

Comparing the terms placed on the both sides of eq. (11) before expressions $[x - \bar{x} + \xi(t)]^i / i!$, $i = 0, 1, 2, \dots$, we get:

$$A_i'(t) + \xi'(t) A_{i+1}(t) = a A_{i+2}(t) \quad i = 0, 1, 2, \dots \tag{12}$$

From conditions (4), (5), and (2) it results that:

$$A_0(t) = T^* \tag{13}$$

$$A_1(t) = -\frac{\gamma K}{\lambda} \xi'(t) \tag{14}$$

Having coefficients $A_0(t)$ and $A_1(t)$ we can determine the remaining coefficients $A_i(t)$, $i = 2, 3, \dots$, by using formula (12). We have:

$$A_{i+2}(t) = \frac{1}{a} [A_i'(t) + \xi'(t) A_{i+1}(t)], \quad i = 0, 1, 2, \dots \tag{15}$$

From this it follows that:

$$A_2(t) = -\frac{\gamma K}{a \lambda} [\xi'(t)]^2 \tag{16}$$

$$A_3(t) = -\frac{\gamma K}{a \lambda} \xi''(t) - \frac{\gamma K}{a^2 \lambda} [\xi'(t)]^3 \tag{17}$$

and so on.

Except coefficient $A_0(t)$ all the other coefficients $A_i(t)$, $i = 1, 2, 3, \dots$, depend on the still unknown function $\xi(t)$, its powers and derivatives. One can try to determine function ξ with the aid of condition (6):

$$\sum_{i=0}^{\infty} A_i(t) \frac{\xi^i(t)}{i!} = \psi(t) \quad (18)$$

However, eq. (18) is so complicated that calculation of function $\xi(t)$ in this way is impossible. Instead one can determine the approximate solutions by taking, for example, only two or three first terms of the series placed on the left side of eq. (18).

Taking into account only two first terms we get the equation:

$$A_0(t) + A_1(t)\xi(t) = \psi(t) \quad (19)$$

Hence, by substituting for $A_0(t)$ and $A_1(t)$ the values defined by relations (13) and (14) we obtain differential equation:

$$T^* - \frac{\gamma\kappa}{\lambda} \xi'(t)\xi(t) = \psi(t) \quad (20)$$

Assumptions taken at the beginning imply that function $\xi(t)$ must have only non-negative values and must satisfy eq. (3). These pieces of information suffice to determine uniquely function $\xi(t)$ by solving eq. (20), it means:

$$\xi(t) = \sqrt{\frac{2\lambda}{\gamma\kappa} \int_0^t (T^* - \psi(\tau)) d\tau} \quad (21)$$

Similarly, by taking only three first terms from the series placed on the left side of eq. (18) we get the relation

$$A_0(t) + A_1(t)\xi(t) + A_2(t) \frac{\xi^2(t)}{2} = \psi(t) \quad (22)$$

and next the differential equation:

$$T^* - \frac{\gamma\kappa}{\lambda} \xi'(t)\xi(t) - \frac{\gamma\kappa}{a\lambda} \frac{[\xi'(t)\xi(t)]^2}{2} = \psi(t) \quad (23)$$

Solving differential eq. (23) under condition (3) we finally receive:

$$\xi(t) = \sqrt{2a \int_0^t \left\{ \sqrt{1 + \frac{2c}{\kappa} [T^* - \psi(\tau)]} - 1 \right\} d\tau} \quad (24)$$

Although formulas (21) and (24) define only the approximate thickness of solidified layer, they can be successively applied in practise. In particular case when $\psi(t) = T^0 = \text{constants}$, $t \in (0, t^*)$, these formulas take very simple form:

$$\xi(t) = \sqrt{\frac{2\lambda}{\gamma\kappa} (T^* - T^0)t} \quad (25)$$

and

$$\xi(t) = \sqrt{2a \left[\sqrt{1 + \frac{2c}{\kappa} (T^* - T^0)} - 1 \right] t} \quad (26)$$

respectively.

To facilitate more the obtained formulas we can introduce another simplification consisted in the assumption that function describing the freezing front location can be approximate by the broken line

where

$$\varphi(t) = \bar{x} - \bar{\xi}(t)$$

$$\bar{\xi}(t) = x_j + m_j(t - t_j), \quad t \in (t_j, t_{j+1}), \quad j = 0, 1, 2, \dots$$

and

$$t_0 = 0, \quad t_{j+1} > t_j, \quad j = 0, 1, 2, \dots$$

$$x_0 = 0, \quad x_j = m_{j-1}(t_j - t_{j-1}) + x_{j-1}, \quad j = 1, 2, 3, \dots$$

Thanks to this the formulas defining functional coefficients A_i become significantly easier:

$$A_0(t) = T^*, \quad t \in (t_j, t_{j+1}), \quad j = 0, 1, 2, \dots$$

$$A_i(t) = -\frac{\gamma \kappa m_j^i}{\lambda a^{i-1}}, \quad t \in (t_j, t_{j+1}), \quad j = 0, 1, 2, \dots$$

for $i = 1, 2, \dots$. On the grounds of these formulas we can construct function $T(x, t)$ and by using the boundary conditions we can obtain function $\xi(t)$ as well. In the above formulas the unknown coefficients $m_j, j = 0, 1, 2, \dots$, appear which can be determined (even analytically) by using the appropriate boundary condition.

Another examples of applying the investigated method and the method in which the freezing front is approximated with the aid of broken line can be found in [24-27].

Numerical method

Results obtained by using the method introduced in the previous section are compared with the results received by applying a numerical method intended for solving the Stefan problem [6, 28].

Let us assume that after $k - 1$ time steps (in moment of time t^{k-1}) the interface is located in node x_{i-1} . Next we accept that after the k -th time step (that is in moment of time t^k) of length Δt^k the interface will be located in node x_i . Condition of temperature continuity implies that in moment t^k the temperature in node x_i will be equal to the solidification temperature T^* . Thus we get the heat conduction problem in interval $[0, x_i]$ with the known boundary conditions of the first kind for $x = 0$ and $x = x_i$, which must be solved. We also know the distribution of temperature in the preceding step of time. In this way, by applying the boundary element method (or other appropriate method) we can determine the temperature distribution in moment t^k and, in consequence, the heat flux on the interface. Determined values depend obviously on the length of time step Δt^k , which is calculated iteratively by using, for this purpose, the Stefan condition expressed in discrete form

$$\gamma \kappa \frac{x_i - x_{i-1}}{\Delta t^k} = -\lambda \frac{\partial T(x_i, t^k)}{\partial x}$$

At the beginning of each calculation stage we assume the initial length of time step which can be, for example, equal to the final length of time step obtained in the previous calculation stage. Next we determine the associated distribution of temperature in moment t^k in region $[0, x_i]$ and on this ground we calculate the heat flux on the interface, it means in point x_i . Using now the Stefan condition we correct the length of time step and we determine again the temperature distribution in moment t^k in region $[0, x_i]$. Procedure of the time step correction is repeated until the required accuracy will be obtained.

In presented calculations we used the boundary element method and the time step was determined iteratively exact to 0.001 s. Most often three or four iterations were enough to obtain the required precision.

Examples

Example 1. Let us assume that the material of solidifying plate of thickness $2\bar{x} = 0.2$ m is characterized by the following parameters: density $\gamma = 7000$ kg/m³, thermal conductivity coefficient $\lambda = 25$ W/mK, specific heat $c = 800$ J/kgK, latent heat of fusion $\kappa = 247$ kJ/kg, solidifi-

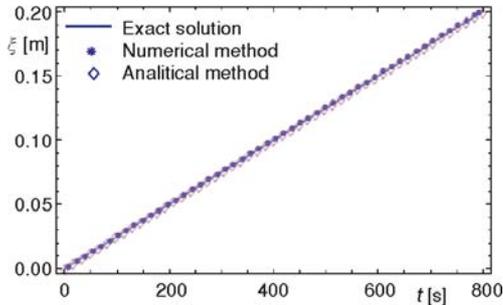


Figure 1. Reconstruction of the solidified layer thickness in example 1

cation temperature $T^* = 1500 \text{ }^\circ\text{C}$, and that the transfer of heat between plate and environment is described by means of condition (6) of the first kind where function ψ has the form:

$$\psi(t) = T^* + \frac{a\kappa}{\lambda} \left[1 - \exp\left(\frac{\bar{x}^2 t}{2 \cdot 10^4 a}\right) \right] \quad (27)$$

Under such assumptions the considered problem possesses the exact solution, which in this case will be the thickness of solidified layer $\xi(t)$ described by relation:

$$\xi(t) = \frac{\bar{x}t}{400} \quad (28)$$

Comparison of reconstruction of the solidified layer thickness, obtained with the aid of both methods presented in previous sections, is displayed in fig. 1. Figure 2 shows the absolute errors of reconstruction of the exact solution.

Obtained results indicate that both discussed methods are efficient – in both cases the absolute errors calculated by using the known form of analytic function $\xi(t)$ describing the sought thickness of solidified layer are very small. Reconstruction of temperature distribution $T(x, t)$ gives similarly satisfying results which additionally confirms the usefulness of considered methods.

In next examples the exact form of sought function ξ is not known, that is why we are not able to calculate the precise errors. Therefore we will only compare the results received by using both methods. Basing on the conclusion from the first example, the similar approximate solutions will be treated as a premise to accept

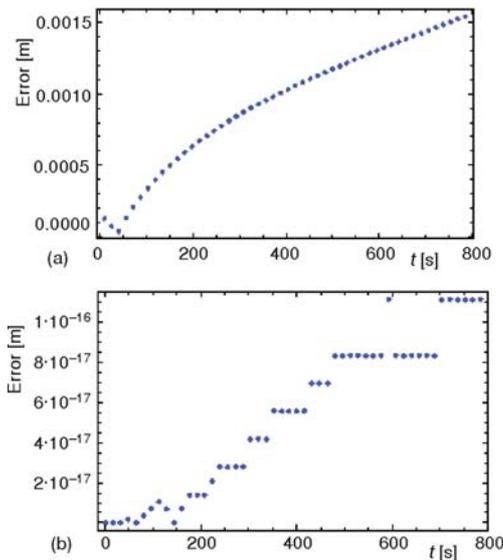


Figure 2. Absolute errors of reconstruction of the solidified layer thickness in example 1; (a) – for numerical method, (b) – for analytic method

both methods to be useful for solving problems of considered kind.

Example 2. Now let us suppose that the material of solidifying plate is characterized by the same values of parameters as in example 1 and that the heat transfer between plate and environment is described by condition (6) of the first kind with function ψ of the form:

$$\psi(t) = 900 \exp\left(\frac{t \ln \frac{4}{9}}{1000}\right) \quad (29)$$

Reconstruction of the thickness of solidified layer calculated by applying both discussed methods is presented in fig. 3. One can observe that obtained approximate solutions cover which allows us to conclude that both reconstructions are correct.

Example 3. Finally we assume that values of parameters describing the material of solidifying plate are again the same as in examples 1 and 2, the heat transfer between plate and environment is characterized by condition (6) of the first kind, but now function ψ is of the following form:

$$\psi(t) = \begin{cases} 1000 - 4t, & \text{for } 0 \leq t < 200, \\ 6t - 1000, & \text{for } 200 \leq t < 300, \\ \frac{7400 - 6t}{7}, & \text{for } 300 \leq t < 1000, \\ 200, & \text{for } t \geq 1000 \end{cases} \quad (30)$$

Comparison of reconstruction of the solidified layer thickness obtained with the aid of analytic and numerical methods is displayed in fig. 4. Since the reconstructions almost cover, which means that the results are very similar, therefore we infer again that both methods can be considered as useful in solving problems of that kind.

Conclusions

In this paper we have presented two methods used for solving the one-phase Stefan problem. First method is of the analytic nature and consists in extension of a function describing the temperature field into the power series, coefficients of which are determined by solving appropriate differential equations constructed by using the boundary conditions. Second method is typically numerical and is based on the boundary element method and special iteration procedure serving for correction of the time step. Three examples presenting results obtained by applying these two procedures show that both methods are useful and effective. In the first example the exact solution of considered problem was known, therefore we could calculate the errors of reconstruction of the solidified layer thickness. Received errors appeared to be very small. On this ground we posed a hypothesis that both methods are efficient which we tried to justify by solving further problems. In two next examples we obtained very similar results which allowed us to conclude, even though we did not know the exact solution, that both discussed methods can be successfully used for solving the one-phase Stefan problem.

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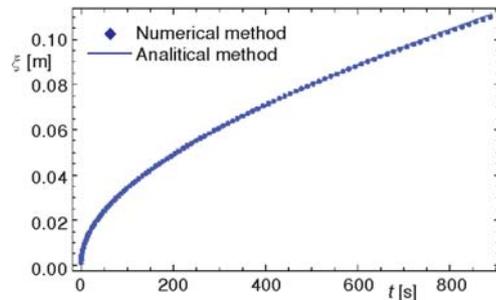


Figure 3. Reconstruction of the solidified layer thickness in example 2

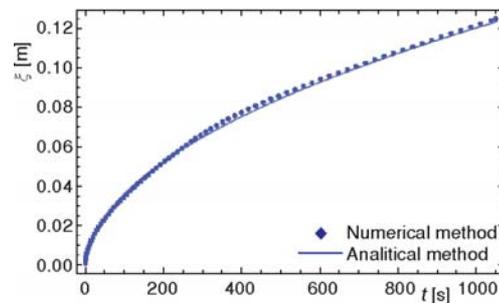


Figure 4. Reconstruction of the solidified layer thickness in example 3

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