CONSTRUCTAL DESIGN OF ISOTHERMAL X-SHAPED CAVITIES

by

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This paper applies constructal design to study the geometry of a X-shaped cavity that penetrates into a solid conducting wall. The objective is to minimize the maximal dimensionless excess of temperature between the solid body and the cavity. There is uniform heat generation on the solid body. The total volume and the cavity volume are fixed, but the geometric lengths and thickness of the X-shaped cavity can vary. The cavity surfaces are isothermal while the solid body has adiabatic conditions on the outer surface. The emerged optimal configurations and performance are reported graphically. When compared to the Y- and C- and H-, the X-shaped cavity performs approximately 53% better than the Y-shaped cavity and 68% better than the C-shaped cavity for the area fraction $\phi = 0.05$, while its performance is 22% inferior to the performance of the H-shaped cavity for the area fraction $\phi = 0.1$. The results indicate that the increase of the complexity of the cavity geometry can facilitate the access of heat currents and improve the performance of the cavities.

Key words: *constructal design, enhanced heat transfer, cavities*

Introduction

Constructal theory is the view that the generation of flow configurations is a physics phenomenon that can be based on a physics principle (the constructal law). The constructal law states that for a finite-size flow system to persist in time (to live), its configuration must evolve in such a way that it provides easier access to the currents that flow through it [1-4].

According to Bejan and Zane [1], everything that moves, whether animate or inanimate, is a flow system. All flow systems generate shape and structure in time in order to facilitate this movement across a landscape filled with resistance. In other words, the designs seen in nature are not the result of chance. They arise naturally, spontaneously, because they enhance access to flow in time.

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In this sense, constructal theory has been used to explain deterministically how configurations in nature has been naturally generated, from inanimate rivers to animate designs, such as vascular tissues, locomotion, and social organization [1]. Chief examples of unifying design are vascular tree-shaped flow architectures, which serve as basis for many rules of animal design [5, 6] and river basin design [2, 7]. This same principle is also used to yields new designs for electronics, fuel cells, and tree networks for transport of people, goods and information (constructal design) [8]. The applicability of this law to the physics of engineered flow systems has been widely discussed in recent literature [9-13].

Among the engineering problems, the field of heat transfer has demonstrated for many years how the principle of generating flow geometry works. The oldest and most clear illustrations are the optimization of solid wall features known as extended surfaces, or fins. More recently, great attention has been devoted to the study of fins array due to its importance in the enhancement of heat transfer in many engineering applications [14-16] such as heat exchangers, internal combustion engines and electric motors. On the other side, open cavities are the regions formed between adjacent fins: if the optimization of the geometry of the individual fin is an important issue, then, certainly, the geometry of the interstices must also be important.

The present numerical study has the purpose to discover, by means of constructal design, the geometrical optimization of a X-shaped cavity that penetrates into a rectangular solid body with uniform internal heat generation. The cavity surfaces are isothermal with a minimum temperature. In this paper, the purpose is to minimize the global thermal resistance between the solid body and the cavity. The ratio between the length of the branches and the length of the stem (L_1/L_0), as well as, the ratio between the thicknesses of the branches are optimized for a fixed degree of freedom H/L = 1.0 and for several values of the ratio between the volume of the cavities and volume of the solid (ϕ).

Mathematical model

Consider the conducting body shown in fig. 1. The configuration is two-dimensional. There is a X-shaped cavity cooling the body of thermal conductivity k. The body generates heat uniformly at the volumetric rate q'''. The outer surfaces of the solid are perfectly insulated. The generated heat current ($q'''A$) is removed by the isothermal cavity at temperature T_0.

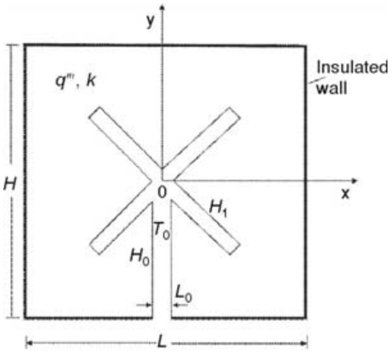

Figure 1. X-shaped cavity

The objective of the analysis is to determine the optimal geometry (L_1/L_0, D_1/D_0) that emerges by minimizing the global thermal resistance $(T_{max} - T_0)/(q'''A/k)$.

According to constructal design, this optimization can be subjected to two constraints, namely, the total area constraint:

$$A = HL \tag{1}$$

and the area occupied by the X-cavity:

$$A_c = 4D_1L_1 + D_1^2 + D_0L_0 + \frac{D_0^2}{4} \tag{2}$$

Equations (1) and (2) can be expressed as the area fraction:

$$\phi = \frac{A_c}{A} \tag{3}$$

Note that there is another geometric constraint given by:

$$\frac{H}{2} = L_0 + \frac{D_0}{2} + \cos\left(\frac{\pi}{4}\right) D_1 \tag{4}$$

The analysis that delivers the dimensionless excess of temperature as a function of the geometry consists of solving numerically the heat conduction equation along the conductivity k-region:

$$\frac{\partial^2 \theta}{\partial \tilde{x}^2} + \frac{\partial^2 \theta}{\partial \tilde{y}^2} + 1 = 0 \tag{5}$$

where the dimensionless variables are:

$$\theta = \frac{T - T_0}{q''' \dfrac{A}{k}} \tag{6}$$

and

$$\tilde{x}, \tilde{y}, \tilde{L}, \tilde{L}_1, \tilde{L}_0, \tilde{D}_2, \tilde{D}_0 = \frac{x, y, H, L, L_1, L_0, D_1, D_0}{\sqrt{A}} \tag{7}$$

The outer surfaces are insulated and their boundary conditions are:

$$\frac{\partial \theta}{\partial \tilde{n}} = 0 \tag{8}$$

The boundary condition in the cavity surfaces is given by an isothermal temperature:

$$\theta_0 = 0 \tag{9}$$

The dimensionless form of eqs. (1), (3), and (4) are:

$$1 = \tilde{L}, \tilde{H} \tag{10}$$

$$\phi = 4\tilde{D}_1 \tilde{L}_1 + \tilde{D}_1^2 + \tilde{D}_0 \tilde{L}_0 + \frac{\tilde{D}_0^2}{4} \tag{11}$$

$$\frac{\tilde{H}}{2} = \tilde{L}_0 + \frac{\tilde{D}_0}{2} + \cos\left(\frac{\pi}{4}\right) \tilde{D}_1 \tag{12}$$

The dimensionless maximal excess of temperature, θ_{max}, is our objective function and is defined as:

$$\theta_{max} = \frac{T_{max} - T_0}{q''' \dfrac{A}{k}} \tag{13}$$

Numerical model

The function defined by eq. (13) can be determined numerically, by solving eq. (5) for the temperature field in every assumed configuration (D_0/L_0, D_1/L_1), and calculating θ_{max} to see whether θ_{max} can be minimized by varying the configuration. In this sense, eq. (5) was solved using a finite elements code, based on triangular elements, developed in MATLAB environment, precisely the partial differential equations (PDE) toolbox [17]. The grid was non-uniform in both $\tilde{x}$ and $\tilde{y}$, and varied from one geometry to the next. The appropriate mesh size was determined by

Table 1. Numerical tests showing the achievement of grid independence (ϕ = 0.1, D_1/D_0 = 0.5, L_1/L_0 = 1.412)

Number of elements	$\theta^j_{\max}$	$\lvert(\theta^j_{\max} - \theta^{j+1}_{\max}) / \theta^j_{\max}\rvert$
440	0.033488	
1760	0.033944	0.01363
7040	0.033936	0.000233
28160	0.033935	0.000048

successive refinements, increasing the number of elements four times from the current mesh size to the next mesh size, until the criterion $\lvert(\theta^j_{\max} - \theta^{j+1}_{\max})/\theta^j_{\max}\rvert < 1\cdot10^{-4}$ was satisfied. Here $\theta^j_{\max}$ represents the maximum temperature calculated using the current mesh size, and $\theta^{j+1}_{\max}$ corresponds to the maximum temperature using the next mesh, where the number of elements was increased by four times. Table 1 gives an example of how grid independence was achieved. The following results were performed by using a range between 20.000 and 50.000 triangular elements. The accuracy of the numerical code has already been demonstrate in several works [18, 19] and will not be shown here.

Optimal geometry

The numerical work consisted of determining the temperature field in a large number of configurations of the type shown in fig.1. Figure 2 shows that there is an optimal ratio L_1/L_0 that minimizes the dimensionless maximal excess of temperature when the degrees of freedom (H/L, D_1/D_0) and the area fraction ϕ are fixed. The results of fig. 2 are summarized in fig. 3. This figure indicates that the once optimized ratio $(L_1/L_0)_o$ increases approximately 10% as the area fraction ϕ increases from ϕ = 0.05 to 0.3. This observation can also be seen in fig. 4. When the area fraction ϕ is small, the X-blades (L_1) penetrate almost completely into the solid body to make easier the optimal distribution of imperfections. Larger L_1 blade decreases its D_1 thickness increasing the resulting L_0 blade and decreasing the ratio L_1/L_0. As ϕ increases there is no need of the X-blades penetrating completely into the solid body. Therefore the optimal L_1 decreases increasing the D_1 thickness and decreasing the L_0 blade as well as increasing the ratio L_1/L_0. It is also interesting to notice that the minimum maximal excess of temperature $\theta_{\max\min}$ decreases approximately 24% as the area fraction ϕ increases from ϕ = 0.05 to 0.3. Figures 3 and 4 confirm

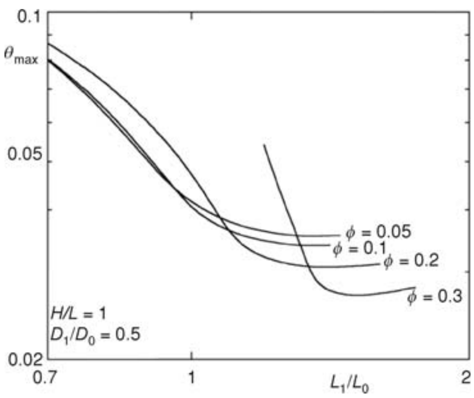

Figure 2. The effect of the area fraction, ϕ, and the ratio between the lengths L_1/L_0 in the maximal dimensionless excess of temperature

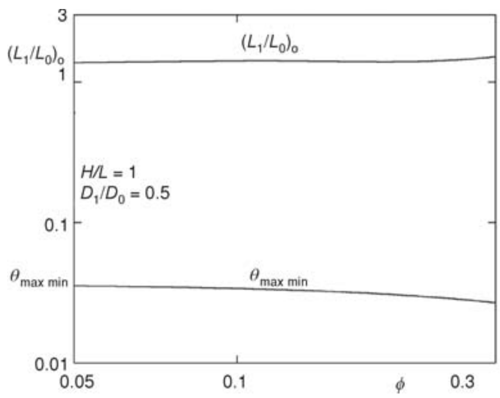

Figure 3. The behavior of the minimal dimensionless excess of temperature, $\theta_{\max\min}$, and the once optimized ratio between the lengths, L_1/L_0, as function of the area fraction ϕ

some features observed in former works, *e. g.* that the cavity performs better when it penetrates almost completely in the solid body. The smaller the area fraction the more the optimal cavity penetrates in the solid body. Figure 4 also illustrates that the hot spots are locate approximately in the same position in all the studied configurations. This result is in agreement with the optimal distribution of imperfections once the hot spots, the maximal dimensionless excess of temperature, can be understood as imperfections in heat transfer systems.

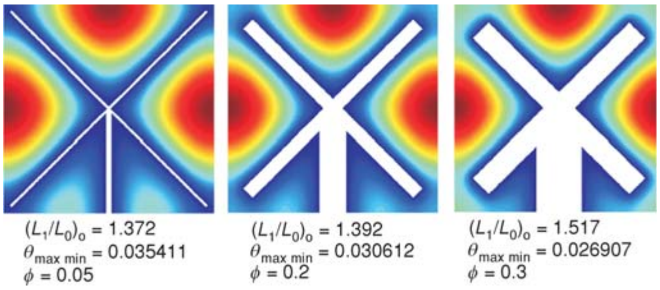

Figure 4. Best shapes calculated in fig. 3
(for color image see journal web site)

The simulations performed in figs. 2 and 3 are repeated using several values of the ratio D_1/D_0. Figure 5 shows the behavior of the maximal dimensionless excess of temperature for several values of the ratio D_1/D_0 as function of the ratio L_1/L_0. This figure confirms that there is a minimum maximal dimensionless excess of temperature and this value decreases as the ratio D_1/D_0 increases. The minimum maximal dimensionless excess of temperature $\theta_{max\ min}$ and the optimal ratio between the lengths $(L_1/L_0)_o$ calculated in fig. 5 are presented as function of the ratio D_1/D_0 in fig. 6. The results show that the effect of the ratio D_1/D_0 in $\theta_{max\,min}$ is not important: $\theta_{max\,min}$ for the ratio $D_1/D_0 = 3$ is only approximately 2% smaller than for the ratio $D_1/D_0 = 0.05$. However, there is a significant effect of the ratio (D_1/D_0) in the optimal ratio $(L_1/L_0)_o$. The ratio $(L_1/L_0)_o$ when $D_1/D_0 = 3$ is approximately 15% smaller than the ratio $(L_1/L_0)_o$ for $D_1/D_0 = 0.5$. The best configurations calculated in fig. 6 are shown in scale in fig. 7. Here we can see how the best configurations evolve when they are free to move. Again the hot spots are optimally distributed for all the calculated best shapes.

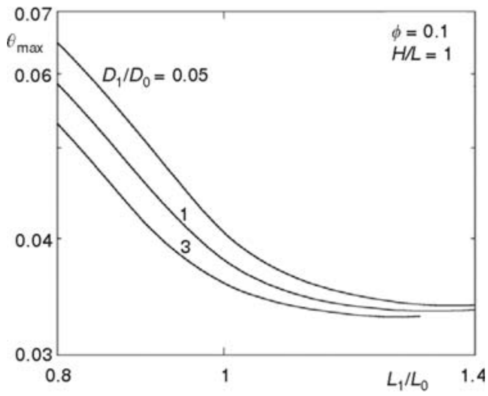

Figure 5. The effect of the ratio between the thicknesses D_1/D_0 and the ratio between the lengths L_1/L_0 in the maximal dimensionless excess of temperature θ_{max}

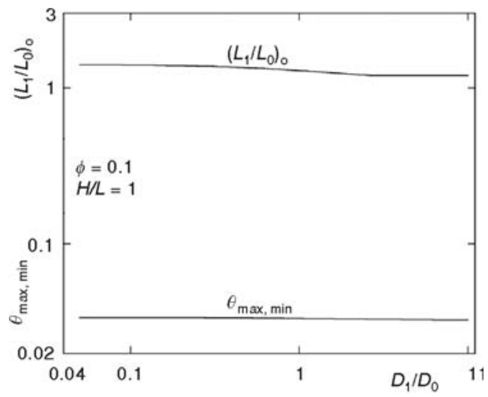

Figure 6. The behavior of the minimal dimensionless excess of temperature $\theta_{max\,min}$ and the once optimized ratio between the lengths $(L_1/L_0)_o$ as function of the ratio between the thicknesses D_1/D_0

Figure 8 shows a comparison among the optimal shapes calculated for the C-, Y-, and X-shaped cavities for the area fraction $\phi = 0.05$. X-shaped cavity performs approximately 53%

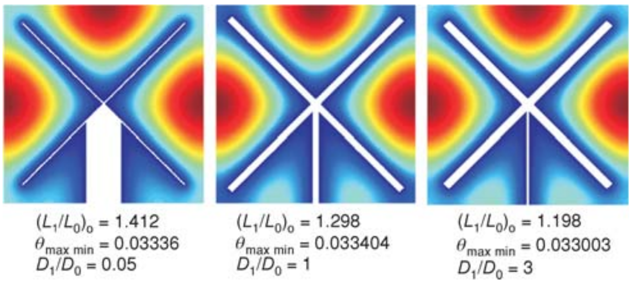

Figure 7. The best shapes calculated in fig. 6
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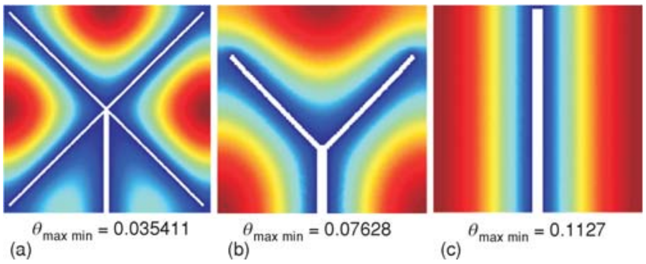

Figure 8. Best shapes (ϕ = 0.05): (a) X-shaped cavity, (b) Y-shaped cavity, (c) C-shaped cavity
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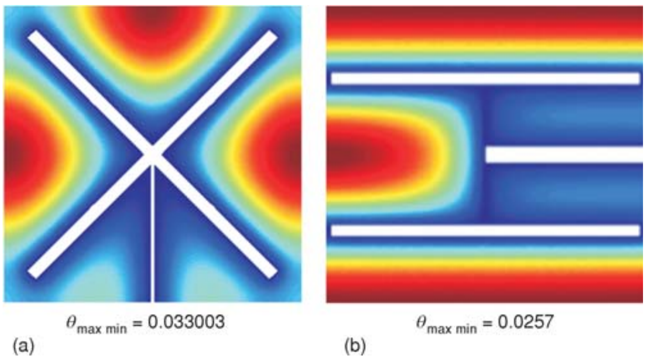

Figure 9. Best shapes (ϕ = 0.1): (a) X-shaped cavity, (b) H-shaped cavity
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better than the Y-shaped cavity and 68% better than the C-shaped cavity. The results indicate that the increase of the complexity of the configuration can improve the performance of the cavity. Figure 9 presents a comparison between X- and H-shaped cavities when the area fraction is $\phi = 0.1$. The results show that the H-shaped cavity performs approximately 22% better than the X-shaped cavity. However, it is important to notice that the H-shaped cavity was optimized for larger degrees of freedom than the X-shaped cavity. Looking at the optimized X-cavity shown in fig. 9 we can observe that we can still optimize it varying the angles of the X. We expect that varying these angles the performance of the X-shaped cavity will improve significantly.

Conclusions

This paper studied numerically a X-shaped cavity cooling a solid body which generates heat uniformly at the volumetric rate q'''. Constructal design is applied to discover the best configurations that facilitate the access of the heat currents. Two degrees of freedom were explored: the ratio between the lengths of the cavity L_1/L_0 and the ratio between the thicknesses of the cavity D_1/D_0. The area fraction ϕ, the ratio between the area of the cavity and the total area, was also a studied parameter. The results show that the performance of the cavity increases approximately 24% as the area fraction increases from 0.05 to 0.3.

The degree of freedom L_1/L_0 has significant effect on the performance of the cavity, *e. g.* the once optimized ratio $(L_1/L_0)_o$ increases approximately 10% as the area fraction ϕ increases from $\phi = 0.05$ to 0.3. However the degree of freedom D_1/D_0 has negligible effect in the performance of the cavity, *e. g.* $\theta_{\max\ \min}$ for the ratio $D_1/D_0 = 3$ is only approximately 2% smaller than for the ratio $D_1/D_0 = 0.05$.

When compared to the Y- and C-, the X-shaped cavity performs approximately 53% better than the Y-shaped cavity and 68% better than the C-shaped cavity for the area fraction $\phi = 0.05$. This indicates that complexity can help to improve the performance of the cavities. In the other side, when compared to the H-shaped cavity the X-shaped cavity has an inferior performance of approximately 22 % when the area fraction $\phi = 0.1$. However, the H-shaped cavity was optimized using larger degrees of freedom (H_2/L_2, L_1/L_2, L_0/L_2, H_1/H_2, H_0/H_2) than the X-shaped cavity (L_1/L_0, D_1/D_0). Therefore, the X-shaped cavity still presents several opportunities of optimization, *e. g.* the angles between the branches of the cavity. This issue will be addressed in future work.

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Nomenclature

A – area, [m^2]
D – thickness, [m]
H – height, [m]
k – thermal conductivity, [Wm^{-1}K^{-1}]
L – length, [m]
q''' – heat uniformly at volumetric rate, [Wm^{-3}]
T – temperature, [K]
x, y – co-ordinates, [m]

Greek symbols

θ – dimensionless temperature (= $(T - T_0)/(q''' A/k)$), [–]
ϕ – area fraction (= A_c/A), [–]

Subscripts

max – maximum
min – minimum
o – once optimized
0 – isothermal wall, single blade
1 – X-blades

Superscript

~ – dimensionless variables

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