# MULTI-WAVE SOLUTIONS FOR A NON-ISOSPECTRAL KDV-TYPE EQUATION WITH VARIABLE COEFFICIENTS

by

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*As a typical mathematical model in fluids and plasmas, Korteweg-de Vries equation is famous. In this paper, the Exp-function method is extended to a nonisospectral Korteweg-de Vries type equation with three variable coefficients, and multi-wave solutions are obtained. It is shown that the Exp-function method combined with appropriate ansatz may provide with a straightforward, effective and alternative method for constructing multi-wave solutions of variable-coefficient non-linear evolution equations.*

Key words: *Korteweg-de Vries type equation, exp-function method, multi-wave solution*

## Introduction

With the development of soliton theory, finding exact solutions of non-linear evolution equations (NLEE) has attached much attention and developed into a significant direction in non-linear science. Since proposed by He and Wu in 2006, the Exp-function method [1] has been applied to many equations, such as the double sine-Gordon equation [2], Maccari's system [3] and variable-coefficient Korteweg-de Vries (KdV) equation [4]. In addition, this method can be generalized for solving differential-difference equation [5], stochastic equation [6], and fractional differential equation [7].

Recently, Zhang [8] generalized the Exp-function method to obtain not only solitary wave solutions and periodic solutions but also rational solutions in a uniform way. Marinakis [9] generalized the Exp-function method and obtained multi-soliton solutions of the famous KdV equation. Based on Marinakis' work, Zhang *et al.* [10] obtained multi-soliton solutions and thus concluded with a uniform formula of the $N$-soliton solution of a KdV equation with two variable coefficients. In 2011, Zhang *et al.* [11] first generalized the Exp-function method for constructing multi-wave solutions of non-linear DDE by devising a rational ansatz of multiple exponential functions.

In this paper, we extend the Exp-function method for constructing multi-wave solutions of a non-isospectral KdV-type equation with variable coefficients $K_0(t)$, $K_1(t)$ and $h(t)$ of time $t$:

$$u_t + K_0(t)(u_{xxx} + 6uu_x) + 4K_1(t)u_x - h(t)(xu_x + 2u) = 0 \tag{1}$$

which was derived by Chan *et al.* from a non-isospectral Lax pair [12].

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## Methodology

We describe the basic idea of the Exp-function method combined with a new and more general ansatz for multi-wave solutions of the given NLEE with variable coefficients, say, in three variables $x$, $y$, and $t$:

$$P(x,y,t,u,u_x,u_y,u_t,u_{xy},u_{xt},u_{yt},u_{xx},u_{yy},u_{tt},\cdots)=0 \tag{2}$$

The Exp-function method for 1-wave solution is based on the assumption that eq. (2) has a solution:

$$u(x,y,t)=\frac{\sum_{i_1=0}^{p_1} a_{i_1}(x,y,t)\,\mathrm{e}^{i_1\xi_1(x,y,t)}}{\sum_{j_1=0}^{q_1} b_{j_1}(x,y,t)\,\mathrm{e}^{j_1\xi_1(x,y,t)}} \tag{3}$$

where $a_{i_1}(x,y,t)$, $b_{j_1}(x,y,t)$, and $\xi_1(x,y,t)$ are unknown functions of the indicated variables, the values of $p_1$ and $q_1$ can be determined by balancing the linear term of highest order in eq. (2) with the highest order non-linear term.

To seek $N$-soliton solutions for integer $N > 1$, we generalize eq. (3) to the form:

$$u(x,y,t)=\frac{\sum_{i_1=0}^{p_1}\sum_{i_2=0}^{p_2}\cdots\sum_{i_N=0}^{p_N} a_{i_1 i_2\cdots i_N}(x,y,t)\,\mathrm{e}^{\sum_{g=1}^{N} i_g\xi_g(x,y,t)}}{\sum_{j_1=0}^{q_1}\sum_{j_2=0}^{q_2}\cdots\sum_{j_N=0}^{q_N} b_{j_1 j_2\cdots j_N}(x,y,t)\,\mathrm{e}^{\sum_{g=1}^{N} i_g\xi_g(x,y,t)}} \tag{4}$$

Substituting eq. (4) with $N = 2$ into eq. (2) and equating to zero each coefficient of the same order power of the exponential functions yields a set of equations. Solving the set of equations, we can determine the 2-wave solution, and the following 3-wave solution by eq. (4) with $N = 3$, provided they exist. If possible, we may conclude with the uniform formula of $N$-wave solution for any $N > 1$.

## Multi-wave solutions

Let us apply the method described in the section *Methodology* to solve the non-isospectral KdV-type equation. To begin with, we suppose that eq. (1) admits 1-wave solution in the form:

$$u(x,t)=\frac{a_1(t)\exp(\xi_1)}{[1+b_1\exp(\xi_1)]^2} \tag{5}$$

where $\xi_1 = k_1(t)x + s_1(t) + w_1$, $k_1(t)$, $s_1(t)$, and $a_1(t)$ are undetermined functions of $t$, $w_1$, and $b_1$ are constants to be determined.

Substituting eq. (5) into eq. (1), and using Mathematica, then equating to zero each coefficient of the same order power of $x^{\theta}\mathrm{e}^{\vartheta\xi_1}\,(\theta=0,1;\ \vartheta=1,2,3,4)$ yields a set of equations for $k_1(t)$, $s_1(t)$, $a_1(t)$ and $b_1$. Solving the set of equations, we have:

$$a_1(t)=2b_1k_{10}^2\mathrm{e}^{2\int h(t)\mathrm{d}t},\ \ k_1(t)=k_{10}\mathrm{e}^{\int h(t)\mathrm{d}t},\ \ s_1(t)=-\int[k_{10}^3K_0(t)\mathrm{e}^{3\int h(t)\mathrm{d}t}+4k_{10}K_1(t)\mathrm{e}^{\int h(t)\mathrm{d}t}]\mathrm{d}t$$

from which we obtain the 1-wave solution of eq. (1):

$$u(x,t)=\frac{2b_1k_{10}^2\mathrm{e}^{2\int h(t)\mathrm{d}t}\mathrm{e}^{\xi_1}}{(1+b_1\mathrm{e}^{\xi_1})^2}=2[\ln(1+b_1\mathrm{e}^{\xi_1})]_{xx} \tag{6}$$

where $\xi_1=xk_{10}\mathrm{e}^{\int h(t)\mathrm{d}t}-\int[k_{10}^3K_0(t)\mathrm{e}^{3\int h(t)\mathrm{d}t}+4k_{10}K_1(t)\mathrm{e}^{\int h(t)\mathrm{d}t}]\mathrm{d}t+w_1$, $b_1$, $k_{10}$, and $w_1$ are arbitrary constants.

We next suppose that eq. (1) has 2-wave solution in the form:

$$u(x,t)=\frac{a_{10}(t)\mathrm{e}^{\xi_1}+a_{01}(t)\mathrm{e}^{\xi_2}+a_{11}(t)\mathrm{e}^{\xi_1+\xi_2}+a_{21}(t)\mathrm{e}^{2\xi_1+\xi_2}+a_{12}(t)\mathrm{e}^{\xi_1+2\xi_2}}{(1+b_1\mathrm{e}^{\xi_1}+b_2\mathrm{e}^{\xi_2}+b_3\mathrm{e}^{\xi_1+\xi_2})^2} \tag{7}$$

where $\xi_1=k_1(t)x+s_1(t)+w_1$, $\xi_2=k_2(t)x+s_2(t)+w_2$, $k_1(t)$, $k_2(t)$, $s_1(t)$, $s_2(t)$, $a_{10}(t)$, $a_{01}(t)$, $a_{11}(t)$, $a_{21}(t)$, $a_{12}(t)$ are undetermined functions of $t$, and $w_1$, $w_2$, $b_1$, $b_2$, and $b_3$ are constants to be determined.

Substituting eq. (7) into eq. (1) and using Mathematica yields:

$$a_{10}(t)=2b_1k_{10}^2\mathrm{e}^{2\int h(t)\mathrm{d}t},\quad a_{01}(t)=2b_2k_{20}^2\mathrm{e}^{2\int h(t)\mathrm{d}t},\quad a_{11}(t)=2b_1b_2\mathrm{e}^{2\int h(t)\mathrm{d}t}(k_{10}-k_{20})^2,$$

$$a_{21}(t)=\frac{2b_1^2b_2k_{20}^2\mathrm{e}^{2\int h(t)\mathrm{d}t}(k_{10}-k_{20})^2}{(k_{10}+k_{20})^2},\ a_{12}(t)=\frac{2b_1b_2^2k_{10}^2\mathrm{e}^{2\int h(t)\mathrm{d}t}(k_{10}-k_{20})^2}{(k_{10}+k_{20})^2},\ b_3=\frac{b_1b_2(k_{10}-k_{20})^2}{(k_{10}+k_{20})^2},$$

$$k_1(t)=k_{10}\mathrm{e}^{\int h(t)\mathrm{d}t},\quad s_1(t)=-\int[k_{10}^3K_0(t)\mathrm{e}^{3\int h(t)\mathrm{d}t}+4k_{10}K_1(t)\mathrm{e}^{\int h(t)\mathrm{d}t}]\mathrm{d}t,$$

$$k_2(t)=k_{20}\mathrm{e}^{\int h(t)\mathrm{d}t},\quad s_2(t)=-\int[k_{20}^3K_0(t)\mathrm{e}^{3\int h(t)\mathrm{d}t}+4k_{20}K_1(t)\mathrm{e}^{\int h(t)\mathrm{d}t}]\mathrm{d}t,$$

from which we obtain 2-wave solution of eq. (1):

$$u(x,t)=2[\ln(1+b_1\mathrm{e}^{\xi_1}+b_2\mathrm{e}^{\xi_2}+b_1b_2\mathrm{e}^{\xi_1+\xi_2+B_{12}})]_{xx} \tag{8}$$

where $\xi_i=xk_{i0}\mathrm{e}^{\int h(t)\mathrm{d}t}-\int[k_{i0}^3K_0(t)\mathrm{e}^{3\int h(t)\mathrm{d}t}+4k_{i0}K_1(t)\mathrm{e}^{\int h(t)\mathrm{d}t}]\mathrm{d}t+w_i$, $b_i$, $k_{i0}$, and $w_i$ are arbitrary constants, $i=1, 2$, and $\mathrm{e}^{B_{12}}=(k_{10}-k_{20})^2/(k_{10}+k_{20})^2$.

Similarly, we can also determine the 3-wave solution of eq. (1):

$$u(x,t)=2[\ln(1+b_1\mathrm{e}^{\xi_1}+b_2\mathrm{e}^{\xi_2}+b_3\mathrm{e}^{\xi_3}+b_1b_2\mathrm{e}^{\xi_1+\xi_2+B_{12}}+$$
$$+b_1b_3\mathrm{e}^{\xi_1+\xi_3+B_{13}}+b_2b_3\mathrm{e}^{\xi_2+\xi_3+B_{23}}+b_1b_2b_3\mathrm{e}^{\xi_1+\xi_2+\xi_3+B_{12}+B_{13}+B_{23}})]_{xx} \tag{9}$$

where $\xi_i=xk_{i0}\mathrm{e}^{\int h(t)\mathrm{d}t}-\int[k_{i0}^3K_0(t)\mathrm{e}^{3\int h(t)\mathrm{d}t}+4k_{i0}K_1(t)\,\mathrm{e}^{\int h(t)\mathrm{d}t}]\mathrm{d}t+w_i$, $b_i$, $k_{i0}$, and $w_i$ are arbitrary constants, $i=1, 2, 3$, and $\mathrm{e}^{B_{ij}}=(k_{i0}-k_{j0})^2/(k_{i0}+k_{j0})^2\,(1\le i<j\le 3)$.

By analyzing the obtained solutions (6), (8) and (9), we can conclude with a uniform formula of $N$-wave solution for any $N>1$ of eq. (1) as:

$$u(x,t)=2[\ln(\sum_{\mu=0,1}\prod_{i=1}^{N}b_i^{\mu_i}\mathrm{e}^{\sum_{i=1}^{N}\mu_i\xi_i+\sum_{1\le i<j\le N}\mu_i\mu_jB_{ij}})]_{xx} \tag{10}$$

where $\xi_i = xk_{i0}\mathrm{e}^{\int h(t)\mathrm{d}t} - \int[k_{i0}^3 K_0(t)\mathrm{e}^{3\int h(t)\mathrm{d}t} + 4k_{i0}K_1(t)\mathrm{e}^{\int h(t)\mathrm{d}t}]\mathrm{d}t + w_i$, $b_i$, $k_{i0}$, $w_i$ are arbitrary constants, the sum $\Sigma_{\mu=0.1}$ refers to all combinations of each $\mu = 0.1$ for $i = 1, 2, \ldots, N$, and $\mathrm{e}^{B_{ij}} = (k_{i0} - k_{j0})^2/(k_{i0} + k_{j0})^2 (1 \le i < j \le N)$.

## Conclusions

In this paper, multi-wave solutions of the nonisospectral KdV-type equation with variable coefficients have successfully been obtained, from which the uniform formula of *N*-wave solution is derived. It is due to the devised new and more general ansatz (4). The paper shows that the Exp-function method combined with appropriate anstaz may provide us with a straightforward, effective and alternative method for constructing multi-wave solutions or testing their existence and can be extended to other NLEE with variable coefficients.

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