A FINITE-DIFFERENCE SCHEME FOR SOLUTION OF A FRACTIONAL HEAT DIFFUSION-WAVE EQUATION WITHOUT INITIAL CONDITIONS

by

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Efficient finite-difference scheme to solve fractional diffusion-wave equations without initial conditions has been developed. The efficient approximation of the Riesz fractional derivatives is demonstrated and efficiently exemplified by two simple problems with/without source terms.

Key words: *fractional diffusion-wave equations, Riesz derivative, finite-difference scheme*

Introduction

Nowadays the fractional differential equations [1, 2] widely encountered in applications to transient rheology [3-5], heat [6-8] and mass transfer [9, 10], non-linear diffusion in porous [10, 11] and granular media [12], and Stefan problem [11-14] are power tools for efficient solutions of complex engineering problems. The fractional differential equations can be solved either analytically by the homotopy perturbation method [15-17], the variational iteration method [11], the Heat-Balance Integral Method [5, 10, 18, 19], the exp-function method [20], and others [21, 22]. The numerical solutions are oriented mainly to finite-difference approximations [23-25], matrix method [26], boundary element method [27], generalized differential transform method [28], and others [29, 30].

The problems without initial solutions are practically important and describe commonly heat diffusion processes far away from the initial start-up moment when the initial conditions do not affect the temperature distributions. A classical example is the temperature distribution in the soil accounting both the daily and annual fluctuations of the temperature at the earth surface. The solutions of such problems allow identifying the thermophysical characteristics of various materials and composition structures forming the grounds having fractal structures with respect to both the time and space co-ordinates.

This work addresses a numerical finite-difference solution the fractional diffusion-wave equation with Riesz derivatives looking for a solution $u(x, y) \in C^2(D)$, in the domain $D = \{0 < x < \infty; 0 < t < T\}$ of a boundary problem described by a non-local heat diffusion equation with Riesz fractional derivatives, namely:

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$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = D\frac{\partial^{\beta} u}{\partial x^{\beta}} + f(x,t) \tag{1a}$$

with boundary conditions:

$$u(0,t) = \mu(t), \quad 0<\alpha\le 1, 1<\beta\le 2, \quad f(x,t),\ \mu(t)\in C(D) \tag{1b, c, d}$$

The Riesz derivatives are defined as [2]:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)\cos\left[\pi\dfrac{1-\alpha}{2}\right]}\int_0^t \frac{u(x,t+s)-u(u,t-s)}{s^{\alpha+1}}\mathrm{d}s \tag{2a}$$

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\beta(\beta-1)}{\Gamma(2-\beta)\cos\left[\pi\dfrac{2-\beta}{2}\right]}\int_0^{+\infty} \frac{u(x+s,t)-2u(x,t)+u(x-s,t)}{s^{\beta+1}}\mathrm{d}s \tag{2b}$$

Solution

Numerical approximation of the fractional derivatives

The problem solution is considered in the domain $\overline{D}^{*} = \{0\le x\le a;\ 0\le t\le \mathrm{T}\}$. From the definition of the Riesz derivative [2] in the time interval $[t_n, t_{n+1}]$, we get:

$$\left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}}\right)_n = \frac{1}{\Gamma(1-\alpha)\cos\left[\pi\dfrac{1-\alpha}{2}\right]}\int_0^{t_n} \frac{u(x,t_n+s)-u(x,t_n-s)}{s^{\alpha+1}}\mathrm{d}s \tag{3a}$$

Expressing $u(x, t+s)$ and $u(x, t-s)$ in Taylor series with respect to the exponent s and then using finite differences for $u'_t(x,t)$ in $[t_n, t_{n+1}]$ as $(\mathrm{d}u/\mathrm{d}t)_n \approx [u(x, t_{n+1}) - u(x, t_{n-1})/2\tau]$ we get a finite-difference approximation of the fractional Riesz derivative of order α in the interval $[t_n, t_{n+1}]$:

$$\left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}}\right)_n \sim \frac{t_n^{1-\alpha}[u(x,t_{n+1})-u(x,t_{n-1})]}{(1-\alpha)\Gamma(1-\alpha)\cos\left[\pi\dfrac{1-\alpha}{2}\right]\cdot\tau} \tag{3b}$$

Similarly, developing $u(x, t+s)$ and $u(x-s,t)$ as Taylor series and representing $u''_{xx}(x,t)$ in the range $[x, x_{m+1}]$ in finite-difference form $(\mathrm{d}^2u/\mathrm{d}x^2)_m \approx [u(x_{m+1}, t) - 2u(x_m, t) + u(x_{m-1}, t)]/h^2$ we get the finite-difference approximation of the fractional derivative of order β as:

$$\left(\frac{\partial^{\beta} u}{\partial t^{\beta}}\right)_m \sim \frac{\beta(\beta-1)\cdot x_m^{2-\beta}[u(x_{m+1},t)-2u(x_m,t)+u(x_{m-1},t)]}{(2-\beta)\Gamma(2-\beta)\cos\left[\pi\dfrac{2-\beta}{2}\right]\cdot h^2} \tag{4}$$

Further, developing the functions $u(x, t_n+\tau)$, $u(x, t_n-\tau)$, $u(x_m+h, t)$, and $u(x_m-h, t)$ as Taylor series (with respect to t and h we get:

$$\frac{t_n^{1-\alpha}[u(x,t_{n+1})-u(x,t_{n-1})]}{(1-\alpha)\Gamma(1-\alpha)\cos\left[\pi\dfrac{1-\alpha}{2}\right]\cdot\tau} \approx \left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}}\right)_n + O(\tau^2) \tag{5a}$$

$$\frac{\beta(\beta-1)\cdot x_m^{2-\beta}[u(x_{m+1},t)-2u(x_m,t)+u(x_{m-1},t)]}{(2-\beta)\Gamma(2-\beta)\cos\left[\pi\frac{2-\beta}{2}\right]\cdot h^2}\approx\left(\frac{\partial^\beta u}{\partial x^\beta}\right)_m+O(h^2) \tag{5b}$$

Therefore, the finite-difference approximations with respect to the time and the space co-ordinate have orders of 2, and 2, respectively.

The numerical solution $\overline{D}^*$ needs a mesh $\Omega = \{(x_{\mathrm{m}}, t_{\mathrm{n}}): n = 0, 1,..., N; m = 0, 1,..., M\}$, where $x_{\mathrm{m}} = mh$, $t_{\mathrm{n}} = n\tau$, $h = a/M$, and $\tau = T/N$. Then, using expressions (3b), (4), and (2) we get a weighted finite-difference scheme:

$$\frac{t_n^{1-\alpha}(u_m^{n+1}-u_m^n)}{(1-\alpha)\Gamma(1-\alpha)\cos\left[\pi\frac{1-\alpha}{2}\right]\cdot\tau}=$$

$$=\frac{\beta(\beta-1)\cdot x_m^{2-\beta}}{(2-\beta)\Gamma(2-\beta)\cos\left[\pi\frac{2-\beta}{2}\right]\cdot h^2}[\sigma\Lambda^\beta u_m^{n+1}+(1-\sigma)u_m^n]+f_m^n \tag{6a}$$

where

$$u_m^{n+1}\approx u(x_m,t_{n+1}),\ u_m^n\approx u(x_m,t_n),\ u_{m-1}^n\approx u(x_{m-1},t_n),\text{ and }u_{m+1}^n\approx u(x_{m+1},t_n) \tag{6b}$$

For $\sigma = 0$ we get an explicit finite-difference:

$$\frac{t_n^{1-\alpha}(u_m^{n+1}-u_m^n)}{\Gamma(2-\alpha)\cos\left[\pi\frac{1-\alpha}{2}\right]\cdot\tau}=\frac{\beta(\beta-1)\cdot x_m^{2-\beta}(u_{m+1}^n-2u_m^n+u_{m-1}^n)}{\Gamma(3-\beta)\cos\left[\pi\frac{2-\beta}{2}\right]\cdot h^2}+f_m^n \tag{7a}$$

Finding the solution on the zero-th layer by the Euler method (8a, b, c):

$$u_0^n=\mu(t_n),\ u_{m+1}^0=u_m^0+hf_{m,0},\ m=0,1,\dots,M-1 \tag{8a,b,c}$$

we can develop the solutions in all other knots of the mesh by applying expression (6a) as it is demonstrated in the next example.

Examples

Example 1

Consider a problem with:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=D\frac{\partial^\beta u}{\partial x^\beta}+\mathrm{e}^{-(x^2+t^2)},\quad u(0,t)=\cos(0{,}7\,t) \tag{9a,b}$$

in the domain $\overline{D}^*=\{0\le x\le 20;\ 0\le t\le 5\}$ by help of expression (6a) and applying expressions (6a) and (8) we get:

$$u_{m+1}^n=\frac{C_{\mathrm t}h^\beta}{C_{\mathrm x}\tau^\alpha}u_m^{n+1}+\left(2-\frac{C_{\mathrm t}h^\beta}{C_{\mathrm x}\tau^\alpha}\right)u_m^n-u_{m-1}^n-\frac{h^\beta}{C_{\mathrm x}}\mathrm{e}^{-(x_m+t_n)} \tag{10a}$$

$$u_0^n=\cos(0.7t_{\mathrm n}),\quad C_{\mathrm t}=\frac{1_n^{1-\alpha}}{\Gamma(2-\alpha)\cos\left[\pi\frac{1-\alpha}{2}\right]},\quad C_x=\frac{x_m^{2-\beta}}{\Gamma(3-\beta)\cos\left[\pi\frac{2-\beta}{2}\right]} \tag{10b,c,d}$$

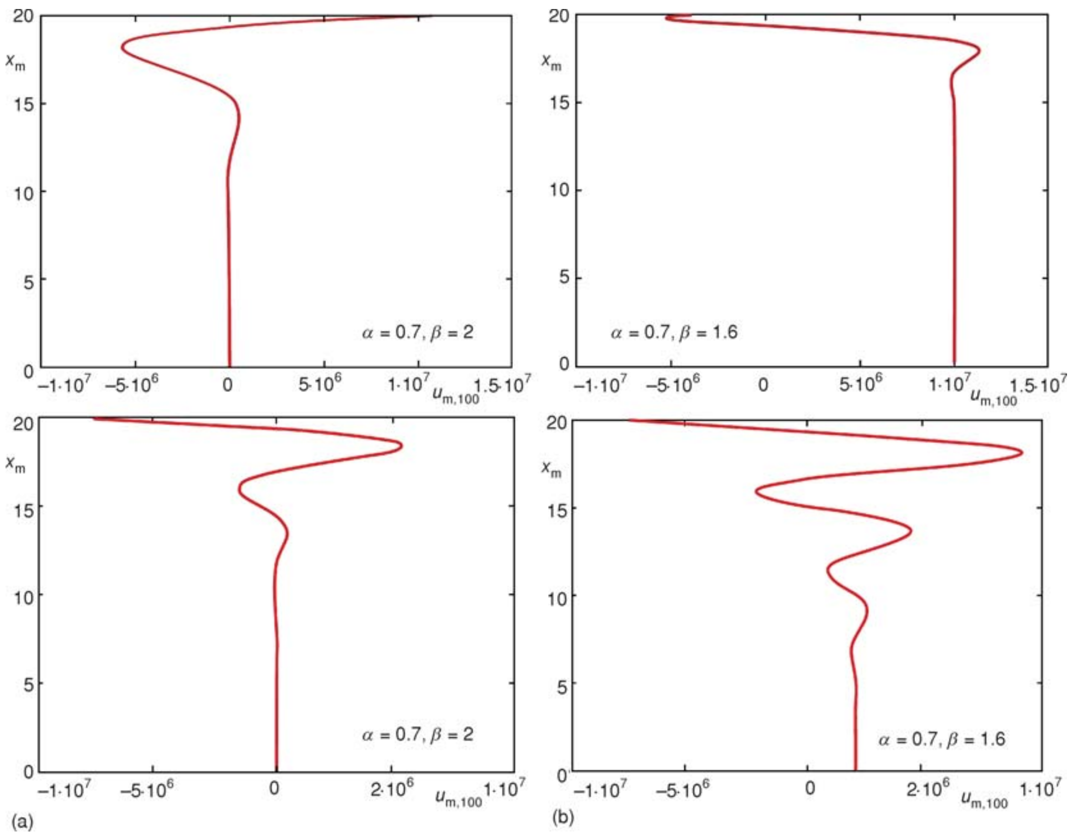

Figure 1. Solutions of the problem 1; (a) for t = 2, (b) for t = 4

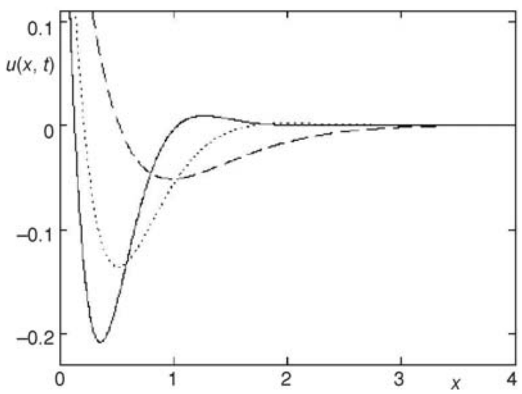

Figure 2. Space distribution of the dimensionless temperature $u(x, t)$ for various values of the fractional order α; solid line – α = 1, dashed line – α = 0.5, dotted line – α = 0.8

This solution is present graphically in fig. 1 for different fractional orders α and β, and times t.

Example 2

Let us solve the problem:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = \cos(0.7t) \quad (11)$$

in the domain $D = \{0 \le x \le 4,\ 0 \le t \le 1\}$.

Applying the same techniques as those to Example 1 we get the results illustrated in fig. 2 at $\alpha < 1$. Plots clearly show that characteristic attenuation length increases while at the same time the characteristic delay time decreases which are reasonable physically sound results.

Conclusions

The fractional calculus allows developing a new approach to the theory of non-local differential equations. Unlike the classical approach where the account of the non-local effects are represented by differential/integral operators are separate terms of the governing equations, the fractional calculus permits those effects to be incorporated in the models by fractional derivatives/integrals. It is quite important to notice that there a large amount of differential models based on the continuum approach and corresponding sets of corresponding multiparametric fundamental solutions. In this context, the fractional calculus formulates new problems, such those that the solution of fractional differential equations create a functional space, a problem still undeveloped in mathematics.

The finite-difference scheme developed in this work and the solutions of the examples based on it show the efficiency of the approach and forms a basis to determine heat diffusivities of heterogeneous media. The present article demonstrates an efficient finite-difference scheme to solve fractional diffusion-wave equations without initial conditions. The efficient approximation of the Riesz fractional derivatives is demonstrated and efficiently exemplified by two simple problems.

The fractional heat diffusion problems considered in the present work allow accounting the time non-locality in a natural way. Such differential equations form a large class of solutions depending on the fractional parameter (order) . For we get the classical solution while in all other case with a non-integer value of the new class of solutions has asymptotic behaviour differing from those of the well-known classical counterparts. It is very important that the functional forms of the new solutions are characterized by the fractional order thus defining a new parameter in the modelling of experimental data.

Nomenclature

a – domain boundary
m – counter of the nodes in the finite-difference scheme, [–]
n – counter of the nodes in the finite-difference scheme, [–]
T – time limit
t – time, [s]
$u(x, t)$ – dimensionless temperature
$u(x, y)$ – dimensionless solution
x – space co-ordinate
y – space co-ordinate

Greek symbols

α – fractional order, [–]
β – fractional order, [–]
Γ – Gamma function

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