

---

**Open forum**

---

**THERMAL IMPEDANCE AT THE INTERFACE OF  
CONTACTING BODIES:  
1-D Example Solved by Semi-Derivatives**

by

**Jordan Hristov**

Department of Chemical Engineering,  
University of Chemical Technology and Metallurgy, Sofia, Bulgaria

Original scientific paper  
DOI: 10.2298/TSCI111125017H

*Simple 1-D semi-infinite heat conduction problems enable to demonstrate the potential of the fractional calculus in determination of transient thermal impedances of two bodies with different initial temperatures contacting at the interface ( $x = 0$ ) at  $t = 0$ . The approach is purely analytic and uses only semi-derivatives (half-time) and semi-integrals in the Riemann-Liouville sense. The example solved clearly reveals that the fractional calculus is more effective in calculation the thermal resistances than the entire domain solutions.*

Key words: *thermal impedance, fractional calculus, half-time fractional derivatives*

**Introduction**

A thermal heating of a half-space is a general academic formulation of high temperature source interaction with materials in the material processing, precision manufacturing, and electronic devices. Although this is a very old subject [1, 2], it remains scientifically important and at the microscopic scale, difficult to master in some configurations [3]. This particularly pertains to transient heat problems with convection, cooling of electronic devices both at the package and system level, and cooling of power semi-conductors using heat sinks [4]. The main problem emerging under such circumstances is in determining the material behavior in transient regime and the heat spreading in depth of the processed materials [2, 5]. In many cases, the time-dependent behaviour relevant to the thermal impedance is of primary importance [5, 6]. Some works on transient temperature field due to heat spots with uniform [3] or continuous or time-dependent heat sources [7] have been developed. The thermal impedance has been estimated by either exact analytical solution [5, 7-9], numerical solutions [5].

The present work demonstrates a straightforward solution to the transient thermal resistance of contacting bodies by half-time fractional derivatives which are not local [10] but al-

---

\* Author's e-mail: jordan.hristov@mail.bg

lowing to express by single relationship the function values and the gradients. Its wide applications to transient rheology [11, 12], heat [6, 9], and mass transfer [13, 14], non-linear diffusion in porous and granular media [15], Stefan problem [16-18] manifest this technique as a power tool for efficient engineering solutions of complex problems.

### Problem statement

Consider the heat diffusion equations of two 1-D bodies with different temperatures prior to the contact (at  $t = 0$ ), namely:

$$\frac{\partial T_1}{\partial t} = \alpha_1 \frac{\partial T_1^2}{\partial x^2}, \quad -\infty < x \leq 0 \quad (1)$$

$$\frac{\partial T_2}{\partial t} = \alpha_2 \frac{\partial T_2^2}{\partial x^2}, \quad 0 < x \leq \infty, \quad 0 < t < \infty \quad (2)$$

with boundary conditions:

$$T_{1\infty} = T_{10}, \quad T_{2\infty} = T_{20}, \quad \lambda_1 \frac{\partial T_1}{\partial x} \Big|_{x=0} = \lambda_2 \frac{\partial T_2}{\partial x} \Big|_{x=0} \quad (3a,b,c)$$

and initial conditions at the interface of mechanical contact:

$$T = T_{10} > 0, \quad x < 0; \quad T = T_{20}, \quad x > 0; \quad T_{10} > T_{20} \quad (3d)$$

We are looking for the surface thermal resistance of both bodies. The thermal impedance of the heated body is defined by the ratio  $Z = [T_2(t)_s - T_{2\infty}] / q_s(t)$ , where  $q_s(t) = -\lambda_2 (\partial T_2 / \partial x) \Big|_{x=+0}$ .

$$Z = \frac{T_2(t)_s - T_{2\infty}}{q_s(t)}, \quad \text{where} \quad q_s(t) = -\lambda_2 \left( \frac{\partial T_2}{\partial x} \right) \Big|_{x=+0} \quad (4a, b)$$

The method developed in the next section demonstrates how using fractional calculus both the nominator and the denominator of (4a) can be determined (at  $x = 0$  without developing entire domain analytical solutions).

### Solutions

#### Fractional calculus approach

Transforming the variables as  $\theta_1 = T_{10} - T_1$  the heat transfer eq. (1) for the hot body is transformed into:

$$\frac{\partial \theta_1}{\partial t} = \alpha_1 \frac{\partial \theta_1^2}{\partial x^2}, \quad \theta(x = -\infty) = 0, \quad \theta(t = 0) = 0, \quad x < 0 \quad (5)$$

At the surface  $x = 0$  the temperature and the heat flux are related as [9, 19]  $(\partial \theta_1 / \partial x) \Big|_{x=0} = (1/\alpha_1^{1/2}) D_t^{1/2} \theta_1$  and substituting  $\theta_1 = T_{10} - T_1$  we have:

$$\frac{\partial T_1}{\partial x} \Big|_{x=0} = \frac{1}{\sqrt{\alpha_1}} \left( D_t^{1/2} T_s - \frac{T_{10}}{\sqrt{\pi t}} \right), \quad x = -0, \quad q_{1(x=0)} = -\lambda_1 \frac{\partial T_1}{\partial x} \Big|_{x=0} \quad (6a,b)$$

Similarly, for the cold body ( $x > 0$ ,  $T_\infty = T_{20}$ ), and  $\theta_2 = T_2 - T_{20}$ , we have:

$$\left. \frac{\partial T_2}{\partial x} \right|_{x=-0} = \frac{1}{\sqrt{\alpha_2}} \left( D_t^{1/2} T_s - \frac{T_{10}}{\sqrt{\pi t}} \right), \quad q_{2(x+0)} = -\lambda_2 \left. \frac{\partial T_2}{\partial x} \right|_{x=+0} \quad (7)$$

Equating the heat fluxes of both bodies at  $x = 0$  and integrating with respect to the time  $t$ , we get:

$$D_t^{1/2} T_s = \frac{1}{\sqrt{\pi t}} \frac{\lambda_1 \sqrt{\alpha_2} T_{10} - \lambda_2 \sqrt{\alpha_1} T_{20}}{\lambda_1 \sqrt{\alpha_2} + \lambda_2 \sqrt{\alpha_1}}; \quad T_s = \frac{\lambda_1 \sqrt{\alpha_2} T_{10} - \lambda_2 \sqrt{\alpha_1} T_{20}}{\lambda_1 \sqrt{\alpha_2} + \lambda_2 \sqrt{\alpha_1}} \quad (8a,b)$$

If the initial temperature of the cold body is assumed zero, *i. e.*,  $T_{20} = 0$  that is equivalent to a shift of the temperature scale origin, then we get the result of Babenko [19]:

$$D_t^{1/2} T_s = \frac{T_{10}}{\sqrt{\pi t}} \frac{\lambda_1 \sqrt{\alpha_2}}{\lambda_1 \sqrt{\alpha_2} + \lambda_2 \sqrt{\alpha_1}}; \quad T_s = \frac{\lambda_1 \sqrt{\alpha_2}}{\lambda_1 \sqrt{\alpha_2} + \lambda_2 \sqrt{\alpha_1}} T_{10} \quad (9a,b)$$

For seek for simplicity, we will assume hereafter that  $T_{20} = 0$ . Then, the heat flux across the contact interface is defined by eq. (7), namely:

$$q_s = -\lambda \left. \frac{\partial T_2}{\partial x} \right|_{x=+0} = \frac{\lambda_2}{\sqrt{\alpha_2}} D_t^{1/2} T_s = \frac{T_{10}}{\sqrt{\pi t}} \frac{\lambda_1 \lambda_2}{\lambda_1 \sqrt{\alpha_1} + \lambda_2 \sqrt{\alpha_1}} \quad (10)$$

Then, the thermal impedance of the heated cold body ( $x > 0$ ,  $T_\infty = 0$ ) is  $Z_2 = T_s/q_s = (\pi)^{1/2} (\alpha_2 t)^{1/2} \lambda_2$  or  $Z_2 = (\pi t)^{1/2} / (\varepsilon_2)^{1/2}$ , where  $\varepsilon = (\rho \lambda C_p)^{1/2}$  is the material heat effusivity.

Similarly, with respect to the chilled body ( $x < 0$ ,  $T_{1\infty} = T_0$ ,) we get:

$$q_{s1} = -\lambda_1 \left. \frac{\partial T_1}{\partial x} \right|_{x=-0} = -\frac{\lambda_1}{\sqrt{\alpha_1}} \left( D_t^{1/2} T_s - \frac{T_{10}}{\sqrt{\pi t}} \right) = \frac{\lambda_1 \lambda_2}{\lambda_1 \sqrt{\alpha_2} + \lambda_2 \sqrt{\alpha_1}} \frac{T_{10}}{\sqrt{\pi t}} \quad (11)$$

$$Z_1 = \frac{T_s - T_{1\infty}}{q_{s1}} = \frac{\sqrt{\alpha_1 t}}{\lambda_1} \sqrt{\pi}, \quad Z_1 = \frac{\sqrt{\pi t}}{\sqrt{\varepsilon_1}} \quad (12a, b)$$

### Comparative classical solutions

The exact solution of the temperature field of both bodies is [20]:

$$T_1(x, t) = T_{10} - \frac{T_{10} - T_{20}}{1 + K_1} \operatorname{erfc} \left( -\frac{x}{2\sqrt{\alpha_1 t}} \right), \quad T_2(x, t) = T_{20} - \frac{T_{10} - T_{20}}{1 + K_2} \operatorname{erfc} \left( -\frac{x}{2\sqrt{\alpha_2 t}} \right) \quad (13a, b)$$

where

$$K_1 = \sqrt{\frac{\lambda_1 C_{p1} \rho_1}{\lambda_2 C_{p2} \rho_2}} = \frac{\varepsilon_1}{\varepsilon_2}; \quad K_2 = \frac{1}{K_1} \quad (14a,b)$$

At  $x = 0$  we get:

$$T_1(0, t) = T_{10} - \frac{T_{10} - T_{20}}{1 + K_1}; \quad T_2(0, t) = T_{20} + \frac{T_{10} - T_{20}}{1 + K_2} \quad (15a,b)$$

$$\lambda_1 \frac{\partial T_1(0, t)}{\partial x} = \lambda_1 \frac{T_{10} - T_{20}}{1 + K_1} \left( \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\alpha_1 t}} \right); \quad \lambda_2 \frac{\partial T_2(0, t)}{\partial x} = \lambda_2 \frac{T_{10} - T_{20}}{1 + K_2} \left( -\frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\alpha_2 t}} \right) \quad (16a, b)$$

Hence,

$$Z_1 = \sqrt{\pi} \frac{\sqrt{\alpha_1 t}}{\lambda_1} = \frac{\sqrt{\pi}}{\sqrt{\varepsilon_1}}; \quad Z_2 = \sqrt{\pi} \frac{\sqrt{\alpha_2 t}}{\lambda_2} = \frac{\sqrt{\pi}}{\sqrt{\varepsilon_2}} \quad (17a, b)$$

### Comments

The method developed here shows direct links between two interrelated issues required to calculate the transient thermal impedance of heated bodies: surface temperatures and fluxes. The common approach is to develop the entire domain solutions through time-wasting techniques, special function, *etc.*, which finally reduce to on only pre-factors depending on the imposed boundary conditions when the space co-ordinate equals to zero. The 1-D problem were especially chosen to demonstrate the approach because such problem are classic in the literature and in many cases either exact or approximate solutions exist. Moreover, it can be a good approximation to real physical problems where heat is transfer by short time contacts such as fluidized bed with immersed surfaces, wearing of sliding parts, safety problems with hot machine elements, *etc.* All results were especially presented as products of  $(\alpha t)^{1/2}/\lambda$  [K/Wm<sup>2</sup>] or the effusivity  $\varepsilon = (\rho\lambda C_p)^{1/2}$ , and terms depending on the type of the imposed boundary condition, as in the classical problems employing entire domain solutions. Cases of 1-D bodies with finite lengths can be also solved by the fractional calculus approach but this needs different techniques beyond the scope of the present work.

### Conclusions

The article presents a method for estimation and calculation of transient thermal impedances of 1-D semi-infinite areas (bodies) when they contact  $t = 0$  at the interface  $x = 0$ . The method is based on relationships based on Riemann-Liouville half-time fractional derivatives and integrals relating both the temperature and the heat flux at any point of the domain. The approach is simple and avoids development of entire-domain solutions.

### Nomenclature

$C_p$	– specific heat capacity, [Jkg <sup>-1</sup> ]	$z$	– dummy variable in the definition of the Riemann-Liouville (RL) fractional derivative, [-]
$q_{s1}$	– surface flux density of the hot body, [Wm <sup>-2</sup> ]	${}^{RL}\partial^{1/2}T(x,t)/\partial_+ x^{1/2} =$	
$q_{s2}$	– surface flux density of the cold body, [Wm <sup>-2</sup> ]	$= [1/\Gamma(1/2)]d/dx \int_0^x T(z,t)/(x-z)^{1/2} dz$	– time-fractional semi-derivative of the Riemann-Liouville (RL) sense
$T$	– temperature, [K]	<i>Greek symbols</i>	
$T_s$	– surface temperature (at the contact interface), [K]	$\alpha$	– thermal diffusivity, [m <sup>2</sup> s <sup>-1</sup> ]
$T_{1\infty}, T_{2\infty}$	– temperature of both bodies far a way from the contacting surface (undisturbed temperature fields), [K]	$\Gamma$	– gamma function
$T_{10}, T_{20}$	– boundary conditions at the contacting interface, [K]	$\varepsilon$	– effusivity ( $= \rho\lambda C_p$ ) <sup>1/2</sup> , [Jm <sup>-2</sup> s <sup>0.5</sup> ]
$t$	– time, [s]	$\lambda$	– heat conductivity, [Wm <sup>-1</sup> K <sup>-1</sup> ]
$x$	– space co-ordinate, [m]	$\rho$	– density, [kgm <sup>-3</sup> ]
$Z$	– thermal impedance (see the definition by eq. 4a), [KW <sup>-2</sup> m <sup>-2</sup> ]	<i>Subscripts</i>	
		1	– hot body
		2	– cold body

## References

- [1] Carslaw, H. S., Jaeger, J. C., Conduction of Heat in Solids, Oxford University Press, London, 1959
- [2] Breaux, H. J., Schlegel, P. T., Transient Heating of Thin Plates, *Int. J. Heat Mass Transfer*, 13 (1970), 1, pp. 218-211
- [3] Muzychka, Y. S., Yovanovich, M. M., Culham, J. R., Thermal Spreading Resistance in Compound and Orthotropic Systems, *J. Thermophysics and Heat Transfer*, 18 (2004), 2, pp. 45-51
- [4] Schneider, G. E., Strong, A. B., Yovanovich, M. M., Transient Thermal Response of Two Bodies, Communicating through a Small Circular Contact Area, *Int. J. Heat Mass Transfer*, 20 (1977), 4, pp. 301-308
- [5] Aderghal, N., *et al.*, Analytical and Numerical Calculation of Surface Temperature and Thermal Constriction Resistance in Transient Dynamic Strip Contact, *Appl. Therm. Eng.*, 31 (2011), 8-9, pp. 1527-1535
- [6] Gabano, J.-D., Pointot, T., Fractional Modelling and Identification of Thermal Systems, *Signal Processing*, 91 (2011), 3, pp. 531-541
- [7] Chaudhry, M. A., Zubair, S. M., Some Analytical Solutions of Time-Dependent, Continuously Operating Heat Sources, *Heat Mass Transfer*, 28 (1993), 4, pp. 217-223
- [8] Hou, Z. B., Komanduri, R., General Solutions for Stationary/Moving Plane Heat Source Problems in Manufacturing and Tribology, *Int. J. Heat Mass Transfer*, 43 (2000), 10, pp. 1679-1698
- [9] Agrawal, O. P., Application of Fractional Derivatives in Thermal Analysis of Disk Brakes, *Nonlinear Dynamics*, 38 (2004), 1-4, pp. 191-206
- [10] Oldham, K. B., Spanier, J., The Fractional Calculus, Academic Press, New York, USA, 1974
- [11] Siddique, I., Vieru, D., Stokes Flows of a Newtonian Fluid with Fractional Derivatives and Slip at the Wall, *Int. Rev. Chem. Eng.*, 3 (2011), 6, pp. 822-826
- [12] Qi, H., Xu, M., Some Unsteady Unidirectional Flows of a Generalized Oldroyd-B Fluid with Fractional Derivative, *Appl. Math. Model.*, 33 (2009), 11, pp. 4184-4191
- [13] dos Santos, M. C., *et al.*, Development of Heavy Metal Orption Isotherm Using Fractional Calculus, *Int. Rev. Chem. Eng.*, 3 (2011), 6, pp. 814-817
- [14] Hristov, J., Starting Radial Subdiffusion from a Central Point through a Diverging Medium (a Sphere): Heat-balance Integral Method, *Thermal Science*, 15 (2011), Suppl. 1, pp. S5-S20
- [15] Pfaffenzeller, R. A., Lenzi, M. K., Lenzi, E. K., Modeling of Granular Material Mixing Using Fractional Calculus, *Int. Rev. Chem. Eng.*, 3 (2011), 6, pp. 818-821
- [16] Meilanov, R. P., Shabanova, M. R., Akhmedov, E. N., A Research Note on a Solution of Stefan Problem with Fractional Time and Space Derivatives, *Int. Rev. Chem. Eng.*, 3 (2011), 6, pp. 810-813
- [17] Voller, V. R., An Exact Solution of a Limit Case Stefan Problem Governed by a Fractional Diffusion Equation, *Int. J. Heat Mass Transfer*, 53 (2010), 23-24, pp. 5622-5625
- [18] Liu, J., Xu, M., Some Exact Solutions to Stefan Problems with Fractional Differential Equations, *J. Math. Anal. Appl.*, 351 (2010), 2, pp. 536-542
- [19] Babenko, Yu. I., Heat-Mass Transfer: Methods for Calculation of Thermal and Diffusional Fluxes (in Russian), Khimia Publ., Moscow, 1984
- [20] Sazonov, V. S., Exact Solution of the Problem of Nonstationary Heat Conduction for two Semi-Spaces in Non-ideal Contact, *J. Eng. Phys. Thermophys.*, 79 (2006), 5, pp. 86-87