TWO DOMINANT ANALYTICAL METHODS FOR THERMAL ANALYSIS OF CONVECTIVE STEP FIN WITH VARIABLE THERMAL CONDUCTIVITY

by

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Heat transfer in a straight fin with a step change in thickness and variable thermal conductivity which is losing heat by convection to its surroundings is developed via differential transformation method, and variational iteration method. In this study, we compare differential transformation method and variational iteration method results, with those of homotopy perturbation method and an accurate numerical solution to verify the accuracy of the proposed methods. As an important result, it is depicted that the differential transformation method results are more accurate in comparison with those obtained by variational iteration method and homotopy perturbation method. After these verifications the effects of parameters such as thickness ratio, ratio, dimensionless fin semi thickness, length ratio, thermal conductivity parameter, and Biot number, on the temperature distribution are illustrated and explained.

Key words: *convective step straight fin, variable thermal conductivity, differential transformation method, variational iteration method, numerical solution*

Introduction

Extended surfaces are used to augment the rate of heat transfer from the primary surface and its convective, radiative or convective-radiative environment in a large variety of thermal equipment. Fins are extensively used in various industrial applications such as air conditioning, refrigeration, automobile, and chemical processing equipment. Krause *et al.* [1] presented a monograph regarding the applications and thermal analyses of fins. They have documented and demonstrated that considering constant thermophysical properties allows scientist to find exact analytical solution for number of cases. It is well-known that existence of large temperature difference within a fin necessitates variable thermal conductivity with temperature. This fact consequently includes one non-linearity within the energy equation of the studied system. From the available published work, about heat transfer in extended surfaces, the following works are of immediate relevance to the present paper. Different configurations of straight fins have been analyzed by Sharqawy and Zubair [2]. They focused on temperature and efficiency within the studied fins and considered both heat and mass transfer mechanisms. The well-known differential transformation method (DTM) was applied to steady-state energy equation within

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triangular fins with constant properties [3]. In an interesting study, Kundu [4] opted in favor of thermal analysis and optimization of longitudinal and pin fins of uniform thickness. The analytical study was carried out for three conditions: fully wet, partially wet and fully dry surfaces. Homotopy analysis method (HAM) was also employed by Domairry and Fazeli [5], to tackle non-linear energy equation in straight fins. They have performed thermal and efficiency analyses with different thermophysical parameters. Considering temperature-dependent thermal conductivity, Arslanturk [6] developed correlation equations for the optimum design of annular fins. Residual minimization technique has been also attracted attention of researchers regarding the analysis of non-linear differential energy equations [7]. Kulkarni and Joglekar [7] used this method to overcome the difficulties within the solution procedure for temperature distribution in a straight convective fin having temperature-dependent thermal conductivity. HAM has been used to derive approximate solutions for the temperature distribution and efficiency of convective fins with simultaneous variation of thermal conductivity and heat transfer coefficient with temperature [8]. Joneidi *et al.* [9] studied an analytical solution of fin efficiency of convective straight fins with temperature-dependent thermal conductivity by the DTM. Variational iteration method (VIM) is another approximate technique which has gained considerable attention in this area. This method can overcome many inherent limitations arising within energy equations such as uncontrollability to the non-zero endpoint boundary conditions. Fouladi *et al.* [10] utilized VIM to solve some examples in the field of heat transfer. Assuming temperature-dependent profile for both thermal conductivity and heat transfer coefficient, Khani and Aziz [11] used HAM to develop an analytical solution for the thermal performance of a straight fin of trapezoidal profile. Ganji *et al.* [12] studied the temperature distribution in an annular fin with temperature dependent thermal conductivity using homotopy perturbation method (HPM). Torabi *et al*. [13] solved the energy equation in the convective-radiative moving fin with variable thermal conductivity using the DTM. They assumed non-zero convection and radiation sink temperature for their analysis.

All of the aforesaid studies are related to fins with constant cross-sectional area or tapered fins. Regarding fins' profile with a step change in cross-sectional area, there are limited studies [14-19]. A pioneering work in this area was done by Aziz [14]. Following his work, Kundu and Das [15] adopted a similar profile for radial fins. Differential quadrature method was used with Malekzadeh *et al.* [16] to optimize convective-radiative flat and step fins. Recently, an annular fin with a step change in thickness was analyzed by Kundu [17]. Both fully and partially wet surface conditions were considered within the analyses. The optimization study was performed and an interesting conclusion was given. It was observed that an annular fin with a step change in thickness is the better choice for the transferring rate of heat in comparison with the concentric-annular disc fin for the same fin volume and identical surface conditions [17]. Kundu and Wongwises [18] applied Adomian decomposition method on the problem of straight find with variable thermal conductivity and heat transfer coefficient.

A careful assessment of the foregoing literature shows that there is just one paper that investigated a problem of convective step fin analytically [19]. The primary purpose of the present paper is to demonstrate the usefulness of DTM and VIM to solve problem of convective heat transfer from a step fin with temperature dependent thermal conductivity. In recently published paper we extended our analyses on the thermal processes of the fin with *convective-radiative* heat transfer and step change in thickness, using differential transformation method (DTM).* Thermal

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analysis of step fins is a new application for DTM and VIM which were used for other engineering applications [20-23]. The results to be presented will highlight the effects of the thickness ratio, α, dimensionless fin semi thickness, δ, length ratio, λ, thermal conductivity parameter, β, and Biot number, Bi, on the temperature distribution.

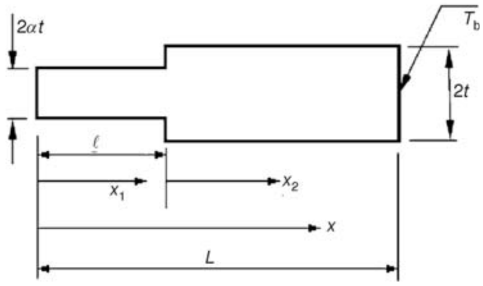

Figure 1. Schematic of a convective fin with a step change in thickness

Description of the problem

A rectangular step fin of unreduced thickness $2t$ and length L is shown in fig. 1. Both surfaces of the fin are convecting to its surroundings. The fin has temperature dependent thermal conductivity k. The base temperature T_b of the fin is constant, and the fin tip is insulated. Since the fin is assumed to be thin, the temperature distribution within the fin does not depend on vertical direction.

The energy balance equation for a differential element of the fin is given as:

$$2\alpha t \frac{\mathrm{d}}{\mathrm{d}x_1}\left[k(T_1)\frac{\mathrm{d}T_1}{\mathrm{d}x_1}\right] - 2h(T_1 - T_\infty) = 0 \tag{1a}$$

$$2t \frac{\mathrm{d}}{\mathrm{d}x_2}\left[k(T_2)\frac{\mathrm{d}T_2}{\mathrm{d}x_2}\right] - 2h(T_2 - T_\infty) = 0 \tag{1b}$$

where $k(T)$ and h are thermal conductivity and heat transfer coefficients, respectively. The thermal conductivity of the fin material is assumed to be a linear function of temperature according to:

$$k(T) = k_0(1 + \kappa T) \tag{2}$$

where k_0 is the thermal conductivity at the base temperature, and κ is the slope of the thermal conductivity-temperature curve. Invoking the continuity of temperature and heat current at the junction, boundary conditions of the governing equations can be expressed as:

$$\begin{gathered} \left.\frac{\mathrm{d}T_1}{\mathrm{d}x_1}\right|_{x_1=0} = 0, \quad T_1(\ell) = T_2(0), \\ \left(k\frac{\mathrm{d}T_2}{\mathrm{d}x_2}\right)_{x_2=0} - [(1-\alpha)h(T_2 - T_4)]_{x_2=0} = \left(\alpha k \frac{\mathrm{d}T_1}{\mathrm{d}x_1}\right)_{x_1=\ell}, \quad T_2(L-\ell) = T_b \end{gathered} \tag{3}$$

Introducing the following dimensionless parameters:

$$\begin{gathered} \theta = \frac{T_1 - T_\infty}{T_b - T_\infty}, \quad \phi = \frac{T_2 - T_\infty}{T_b - T_\infty}, \quad \zeta = \frac{x}{L}, \quad \xi = \frac{x_1}{L}, \quad \tau = \frac{x_2}{L}, \quad \lambda = \frac{\ell}{L} \\ \delta = \frac{t}{L}, \quad \beta = \kappa(T_b - T_\infty), \quad \mathrm{Bi} = \frac{hL}{k_0}, \quad \Psi^2 = \frac{\mathrm{Bi}}{\alpha\delta}, \quad \Omega^2 = \frac{\mathrm{Bi}}{\delta} \end{gathered} \tag{4}$$

The formulation of the fin problem reduces to the following equation:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} + \beta\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} + \beta\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)^2 - \Psi^2\theta = 0, \quad 0 \le \xi \le \lambda \tag{5a}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}+\beta\phi\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}+\beta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2-\Omega^2\phi=0,\quad 0\le\tau\le 1-\lambda \tag{5b}$$

with the following boundary conditions:

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_{\xi=0}=0 \tag{6a}$$

$$\theta(\lambda)=\phi(0) \tag{6b}$$

$$\left[(1+\beta\phi)\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right]_{\tau=0}-[(1-\alpha)\mathrm{Bi}\phi]_{\tau=0}=\left[\alpha(1+\beta\theta)\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right]_{\xi=\lambda} \tag{6c}$$

$$\phi(1-\lambda)=1 \tag{6d}$$

Fundamental of differential transformation method [24]

Let $x(t)$ be analytic in a domain D and let $t = t_i$ represent any point in D. The function $x(t)$ is then represented by one power series whose center is located at t_i. The Taylor series expansion function of $x(t)$ is in form of:

$$x(t)=\sum_{k=0}^{\infty}\frac{(t-t_i)^k}{k!}\left[\frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}\right]_{t=t_i}\qquad \forall t\in D \tag{7}$$

The particular case of eq. (7) when $t_i = 0$ is referred to as the Maclaurin series of $x(t)$ and is expressed as:

$$x(t)=\sum_{k=0}^{\infty}\frac{(t)^k}{k!}\left[\frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}\right]_{t=0} \tag{8}$$

As explained in [25] the differential transformation of the function is defined as:

$$X(k)=\sum_{k=0}^{\infty}\frac{(H)^k}{k!}\left[\frac{d^k x(t)}{\mathrm{d}t^k}\right]_{t=0} \tag{9}$$

where $x(t)$ is the original function and $X(k)$ – the transformed function. The differential spectrum of $X(k)$ is confined within the interval $t\in[0, H]$, where H is a constant. The differential inverse transform of $X(k)$ is defined as:

$$x(t)=\sum_{k=0}^{\infty}\left(\frac{t}{H}\right)^k X(k) \tag{10}$$

It is clear that the concept of differential transformation is based upon the Taylor series expansion. The values of function $X(k)$ at values of argument k are referred to as discrete, *i. e.* $X(0)$ is known as the zero discrete, $X(1)$ as the first discrete, *etc*. The more discretes available, the more precise it is possible to restore

Table 1. The fundamental operations of differential transform method

Original function	Transformed function
$x(t)=\alpha f(t)\pm\beta g(t)$	$X(k)=\alpha F(k)\pm\beta G(k)$
$x(t)=\frac{\mathrm{d}f(t)}{\mathrm{d}t}$	$X(k)=(k+1)F(k+1)$
$x(t)=\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}$	$X(k)=(k+1)(k+2)F(k+2)$
$x(t)=t^{\mathrm{m}}$	$X(k)=\delta(k-m)=\begin{cases}1 & k=m\\ 0 & k\ne m\end{cases}$
$x(t)=\exp(\lambda t)$	$X(k)=\frac{\lambda^k}{k!}$
$x(t)=f(t)g(t)$	$X(k)=\sum_{l=0}^{k}F(l)G(k-l)$

the unknown function. The function $x(t)$ consists of T-function $X(k)$, and its value is given by the sum of the T-function with $(t/H)^k$ as its coefficient. In real applications, at the right choice of the constant H, the larger values of argument k, the discrete of spectrum reduce rapidly. The function $x(t)$ is expressed by a finite series and eq. (10) can be written as:

$$x(t)=\sum_{k=0}^{\infty}\left(\frac{t}{H}\right)^k X(k) \tag{11}$$

Mathematical operations performed by differential transform method are listed in tab. 1.

Solution with DTM

Now we apply the DTM into eq. (5a). Taking the differential transform of eq. (5a) with respect to ξ, and considering $H = 1$ according to tab. 1, gives:

$$(k+2)(k+1)\Theta(k+2)+\beta\left[\sum_{l=0}^{k}\Theta(l)(k+2-l)(k+1-l)\Theta(k+2-l)\right]+ \\ +\beta\left[\sum_{l=0}^{k}(l+1)\Theta(l+1)(k+1-l)\Theta(k+1-l)\right]-\Psi^2\Theta(k)=0 \tag{12}$$

From boundary condition in eq. (6a), that we have it at point $\xi = 0$, and exerting transformation:

$$\Theta(1)=0 \tag{13}$$

The other boundary conditions are considered as:

$$\Theta(0)=C_1 \tag{14}$$

Accordingly, from a process of inverse differential transformation, in this problem we calculated $\Theta(k+2)$ from eq. (12) as:

$$\Theta(2)=\frac{1}{2}\frac{\Psi^2 C_1}{1+\beta C_1} \tag{15a}$$

$$\Theta(3)=0 \tag{15b}$$

$$\Theta(4)=-\frac{1}{24}\frac{\Psi^4 C_1(-1+2\beta C_1)}{(1+\beta C_1)^3} \tag{15c}$$

$$\Theta(5)=0 \tag{15d}$$

$$\Theta(6)=\frac{1}{720}\frac{\Psi^6 C_1(-1+2\beta C_1)(-1+14\beta C_1)}{(1+\beta C_1)^5} \tag{15e}$$

$$\Theta(7)=0 \tag{15f}$$

$$\Theta(8)=-\frac{1}{40320}\frac{\Psi^8 C_1(-1+2\beta C_1)(1-76\beta C_1+448\beta^2 C_1^2)}{(1+\beta C_1)^7} \tag{15g}$$

$$\Theta(9)=0 \tag{15h}$$

$$\Theta(10)=-\frac{1}{3628800}\frac{\Psi^{10} C_1(-1+2\beta C_1)(-1+330\beta C_1-7152\beta^2 C_1^2+25592\beta^3 C_1^3)}{(1+\beta C_1)^9} \tag{15i}$$

$$\Theta(11)=0 \tag{15j}$$

This process may be continued further. Substituting eq. (15) into the main equation based on DTM, the closed form of the solutions is obtained as:

$$\theta(\xi)=C_1+\frac{1}{2}\frac{\Psi^2C_1}{1+\beta C_1}\xi^2-\frac{1}{24}\frac{\Psi^4C_1(-1+2\beta C_1)}{(1+\beta C_1)^3}\xi^4+$$
$$+\frac{1}{720}\frac{\Psi^6C_1(-1+2\beta C_1)(-1+14\beta C_1)}{(1+\beta C_1)^5}\xi^6-$$
$$-\frac{1}{40320}\frac{\Psi^8C_1(-1+2\beta C_1)(1-76\beta C_1+448\beta^2C_1^2)}{(1+\beta C_1)^7}\xi^8+$$
$$+\frac{1}{3628800}\frac{\Psi^{10}C_1(-1+2\beta C_1)(-1+330\beta C_1-7152\beta^2C_1^2+25592\beta^2C_1^3)}{(1+\beta C_1)^9}\xi^{10} \tag{16}$$

Also, we apply the DTM into eq. (5b). Taking the differential transform of eq. (5b) with respect to τ, and considering $H = 1$ according to tab. 1, gives:

$$(k+2)(k+1)\Phi(k+2)+\beta\left[\sum_{l=0}^{k}\Phi(l)(k+2-1)(k+1-l)\Phi(k+2-l)\right]+$$
$$+\beta\left[\sum_{l=0}^{k}(l+1)\Phi(l+1)(k+1-l)\Phi(k+1-l)\right]-\Omega^2\Phi(k)=0 \tag{17}$$

Letting $\phi(0) = C_2$ and $(\mathrm{d}\phi/\mathrm{d}t)|_{\tau=0} = C_3$ and exerting transformation

$$\Phi(0)=C_2 \text{ and } \Phi(1)=C_3 \tag{18}$$

Using the same procedure as introduced in eq. (15), the closed form of the solutions is:

$$\phi(\tau)=C_2+C_3\tau+\frac{1}{2}\frac{(-\beta C_3^2+\Omega^2C_2)}{1+\beta C_2}\tau^2-\frac{1}{6}\frac{C_3(-\Omega^2+2\Omega^2\beta C_2-3\beta^2C_3^2)}{(1+\beta C_2)^2}\tau^3-$$
$$-\frac{1}{24}\frac{(5\Omega^2\beta C_3^2-13\Omega^2\beta^2C_2C_3^2-\Omega^4C_2+2\Omega^4\beta C_2^2+15\beta^3C_3^4)}{(1+\beta C_2)^3}\tau^4+\cdots \tag{19}$$

Integration constant C_1 represents the temperature at the fin tip. Here C_2, and C_3 are temperature and temperature gradient at the cross-section where the step change in thickness occurs, respectively. The constants can be evaluated from the boundary conditions given in eqs. (6b-6d). We employed the Maple's built-in the fsolve command which numerically approximates the roots of an algebraic function using the specified method, such as Newton-Raphson. This command uses the Newton-Raphson method by default.

As an example, let us assume $\alpha = 0.5, \lambda = 0.5, \delta = 0.05, \beta = -0.4$, and Bi = 0.01. Therefore, the values of C_1, C_2, and C_3, applying $n = 10$ which will be used in this paper, will be obtained as:

$$C_1 = 0.8273513861, \quad C_2 = 0.8911784779, \quad C_3 = 0.1387529079 \tag{20}$$

Substituting these obtained C_1, C_2, and C_3 parameters in eqs. (18), the temperature profile of fin for this special case will be:

$$\theta(\xi)=0.8273513861+0.2473177508\xi^2+0.0306058589\xi^4+0.0051353090\xi^6+\cdots$$
$$\phi(\tau)=0.8911784779+0.1387529079\tau+0.1444664937\tau^2+0.0196466096\tau^3+ \tag{21}$$
$$+0.0119222332\tau^4+0.0030977250\tau^5+0.001581207\tau^6+\cdots$$

The calculations reported in this paper use $n = 10$ which was found to be sufficient to give an accurate solution. An implication of this is that eq. (5) only requires the summation of a limited number of terms, and therefore the solution can be computed without excessive computational effort.

Fundamental of variational iteration method [26]

To illustrate the basic concept of the technique, we consider the following general differential equation:

$$Lu + Nu = g(x) \tag{22}$$

where L is a linear operator, N a non-linear operator, and $g(x)$ is the forcing term. According to the VIM, we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda[Lu_n(t) + N\tilde{u}_n(t) - g(t)]\mathrm{d}t \tag{23}$$

where λ is a Lagrange multiplier, which can be identified optimally *via* the variational iteration method. The subscripts n denote the n^{th} approximation, $\tilde{u}_n$ is considered as a restricted variation, that is, $\delta\tilde{u}_n = 0$; eq. (23) is called a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. In this method, it is required first to optimally determine the Lagrange multiplier λ. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by:

$$u = \lim_{n\to\infty} u_n \tag{24}$$

Solution with VIM

In order to solve eq. (5a) using the VIM, we construct a correction functional:

$$\theta_{n+1}(\xi) = \theta_n(\xi) + \int_0^\xi \lambda\left\{\frac{\mathrm{d}^2\theta_n}{\mathrm{d}t^2} + \beta\tilde{\theta}_n\frac{\mathrm{d}^2\tilde{\theta}_n}{\mathrm{d}t^2} + \beta\left(\frac{\mathrm{d}^2\tilde{\theta}_n}{\mathrm{d}t}\right)^2 - \Psi^2\tilde{\theta}_n\right\}\mathrm{d}t \tag{25}$$

Taking variation with respect to the independent variable θ_n, noticing that $\delta\tilde{\theta}_n = 0$:

$$\begin{aligned}\delta\theta_{n+1}(\xi) &= \delta\theta_n(\xi) + \delta\int_0^\xi \lambda\left\{\frac{\mathrm{d}^2\theta_n}{\mathrm{d}t^2}\right\}\mathrm{d}t,\\ &= \delta\theta_n(\xi) + \lambda\delta\theta_n'(t)|_{t=\xi} - \lambda'\delta\theta_n(t)|_{t=\xi} + \int_0^\xi(\lambda'')\delta\theta_n\mathrm{d}t = 0\end{aligned} \tag{26}$$

for all variations $\delta\theta_{\text{n}}$ and $\delta\theta_n'$, its stationary conditions can be obtained:

$$\delta\theta_n : \lambda''(t)|_{t=\xi} = 0, \quad \delta\theta_n : 1 - \lambda'(t)|_{t=\xi} = 0, \quad \delta\theta_n' : \lambda(t)|_{t=\xi} = 0 \tag{27}$$

The Lagrangian multiplier can therefore be identified as:

$$\lambda = t - \xi \tag{28}$$

As a result, we obtain the iteration formula:

$$\theta_{n+1}(\xi) = \theta_n(\xi) + \int_0^\xi (\tau - \xi)\left\{\frac{\mathrm{d}^2\theta_n}{\mathrm{d}t^2} + \beta\theta_n\frac{\mathrm{d}^2\tilde{\theta}_n}{\mathrm{d}t^2} + \beta\left(\frac{\mathrm{d}\theta_n}{\mathrm{d}t}\right)^2 - \Psi^2\theta_n\right\}\mathrm{d}t \tag{29}$$

Let $(\mathrm{d}\theta/\mathrm{d}\xi)|_{\xi=0} = 0$ from eq. (6a), together with $\theta(0) = C_1$, an arbitrary initial approximation that satisfies the initial conditions is obtained as:

$$\theta_0(\xi) = C_1 \tag{30}$$

Using the variational formula, eq. (29), we have:

$$\theta_1(\xi) = C_1 + \frac{1}{2}\Psi^2 C_1 \xi^2 \tag{31a}$$

$$\theta_2(\xi) = C_1 + \left(\frac{1}{2}\Psi^2 C_1 - \frac{1}{2}\beta\Psi^2 C_1^2\right)\xi^2 - \frac{1}{12}\left(\frac{3}{2}\beta\Psi^4 C_1^2 - \frac{1}{2}\Psi^4 C_1\right)\xi^4 \tag{31b}$$

Accordingly, in the same manner the rest of the components of the iteration formula can be obtained.

Letting $\phi(0) = C_2$ and $(\mathrm{d}\phi/\mathrm{d}t)|_{t=0} = C_3$ and applying the same procedure to eq. (5b), it can be written as:

$$\phi_0(\tau) = C_2 + C_3\tau \tag{32a}$$

$$\phi_1(\tau) = C_2 + C_3\tau + \frac{1}{6}(-3\beta C_3^2 + 3\Omega^2 C_2)\tau^2 + \frac{1}{6}\Omega^2 C_3\tau^3 \tag{32b}$$

Here, in the wake of the large term of second and third iterations for the solution, the result of the first iterations is shown; however the obtained results are calculated using three iterations.

The constants C_1, C_2, and C_3 can be evaluated from the boundary conditions given in eqs. (6b-6d) using the Newton-Raphson method.

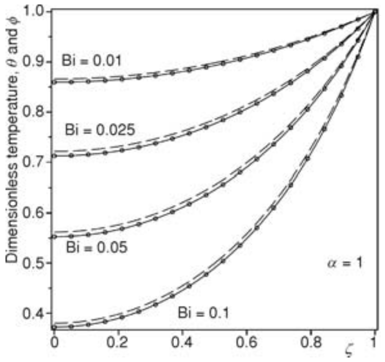

Figure 2. Comparison of dimensionless temperature variation obtained by DTM (solid line), VIM (dashed line), and NS (circle)

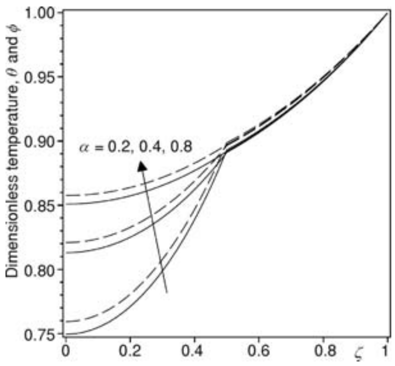

Figure 3. Dimensionless temperature variation obtained by DTM (solid line) and VIM (dashed line) for various values of α

Results and discussion

Two analytical solutions named as the differential transformation and variational iteration methods were applied to eq. (5). Figure 2 indicates that the differences among the DTM, VIM, and the numerical solution (NS) for mentioned equation. For this boundary value problem a finite difference technique with Richardson extrapolation used, which was characterized with the maximum number of 128 points and an absolute error of $1\cdot10^{-6}$. In this figure, we assume that the fin is without step in thickness *i. e.* $\alpha = 1$. Although the VIM results are acceptable, but it is shown that with the DTM, a highly accurate analytical solution of the problem is achievable. Accordingly, in order to investigate the accuracy of the DTM solution with a finite number of terms, the corresponding results are compared with the HPM [19], VIM, and numerical solution by using MAPLE which uses a finite difference method with Richardson extrapolation in tabs. 2 and 3. These tables represent tip temperature and junction temperature, respectively. The results of the comparison clearly show that the maximum difference between HPM and numerical results for tip temperature for the strongest non-linearity condition, *i. e.*, Bi = 0.1 and $\beta = -0.5$, is 0.25%. However, this value for the DTM solution is 0.04%. It should be noted that, for all numerical results reported here, the following values of variables were used unless otherwise indicated by the graphs or tables. $\alpha = 0.5$, $\lambda = 0.5$, $\delta = 0.05$, $\beta = -0.4$, and Bi = 0.01.

Table 2. The results of VIM, HPM, DTM, and their errors for tip temperature

Bi	β	HPM [17]	DTM	VIM	NS	Error HPM [%]	Error DTM [%]	Error VIM [%]
0.01	–0.5	0.80524	0.80483	0.81995	0.80477	0.058402	0.007456	1.886253
	–0.3	0.84579	0.84577	0.84884	0.84573	0.007094	0.004730	0.367730
	–0.1	0.87370	0.87369	0.87382	0.87366	0.004578	0.003434	0.018314
	0	0.88441	0.88441	0.88441	0.88433	0.009046	0.009046	0.009046
	0.1	0.89354	0.89353	0.89348	0.89347	0.007835	0.006715	0.001119
	0.3	0.90821	0.90820	0.90641	0.90816	0.005506	0.004405	0.192700
	0.5	0.91946	0.91943	0.91173	0.91935	0.011965	0.008702	0.828850
0.1	–0.5	0.28338	0.28280	0.29472	0.28266	0.254723	0.049529	4.266610
	–0.3	0.32246	0.32241	0.32654	0.32239	0.021713	0.006204	1.287261
	–0.1	0.36172	0.36171	0.36221	0.36171	0.002765	0	0.138232
	0	0.38097	0.38097	0.38099	0.38096	0.002625	0.002625	0.007875
	0.1	0.39984	0.39983	0.39986	0.39982	0.005002	0.002501	0.010005
	0.3	0.43616	0.43616	0.43521	0.43613	0.006879	0.006879	0.210950
	0.5	0.47029	0.47028	0.46180	0.47024	0.010633	0.008506	1.794830

Table 3. The results of VIM, HPM, DTM, and their errors for junction temperature

Bi	β	HPM [17]	DTM	VIM	NS	Error HPM [%]	Error DTM [%]	Error VIM [%]
0.01	–0.5	0.87556	0.87520	0.88630	0.87513	0.049136	0.007999	1.276382
	–0.3	0.90377	0.90376	0.90588	0.90371	0.006639	0.005533	0.240121
	–0.1	0.92213	0.92213	0.92221	0.92209	0.004338	0.004338	0.013014
	0	0.92900	0.92900	0.92900	0.92892	0.008612	0.008612	0.008612
	0.1	0.93479	0.93478	0.93475	0.93472	0.007489	0.006419	0.003210
	0.3	0.94398	0.94398	0.94285	0.94392	0.006356	0.006356	0.113360
	0.5	0.95096	0.95093	0.94612	0.95085	0.011569	0.008414	0.497450
0.1	–0.5	0.47625	0.47527	0.49301	0.47500	0.263158	0.056842	3.791579
	–0.3	0.52508	0.52500	0.53043	0.52497	0.020954	0.005715	1.040059
	–0.1	0.56845	0.56845	0.56900	0.56843	0.003518	0.003518	0.100276
	0	0.58787	0.58786	0.58789	0.58785	0.003402	0.001701	0.006804
	0.1	0.60588	0.60587	0.60588	0.60585	0.004952	0.003301	0.004952
	0.3	0.63808	0.63808	0.63680	0.63804	0.006269	0.006269	0.194350
	0.5	0.66589	0.66590	0.65691	0.66585	0.006007	0.007509	1.342640

Figure 3 shows the effect of the thickness ratio *i. e.* parameter α on the temperature distribution in the step fin. The bottom curve corresponds to α = 0.2 and the top curve corresponds to α = 0.8. As the parameter α increases, the temperature distribution within the thin section of the fin increases, and the temperature distribution within the thick section of the fin decreases but as expected it is not significant.

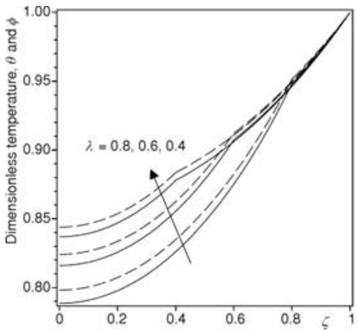

Figure 4. Dimensionless temperature variation obtained by DTM (solid line) and VIM (dashed line) for various values of λ

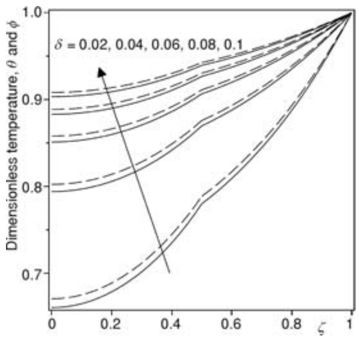

Figure 5. Dimensionless temperature variation obtained by DTM (solid line) and VIM (dashed line) for various values of δ

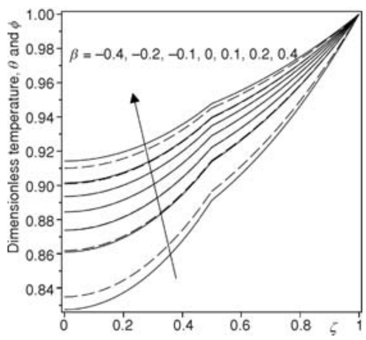

Figure 6. Dimensionless temperature variation obtained by DTM (solid line) and VIM (dashed line) for various values of β

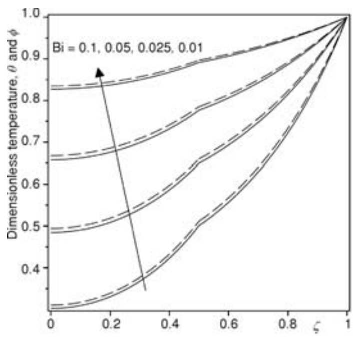

Figure 7. Dimensionless temperature variation obtained by DTM (solid line) and VIM (dashed line) for various values of Bi

The effect of length ratio parameter, *i. e.*, λ, on the temperature distribution within the fin has been plotted in fig. 4. It is clear from this figure that as λ increases, *i. e.*, as the thin section of the fin increases, the temperature distribution within the thin section of the fin decreases.

For the case of different values for dimensionless fin semi thickness, results of the present analysis are depicted in fig. 5. As δ decreases, the cooling becomes more effective, promoting lower temperatures in the fin. This interesting behavior occurs for both thin and thick sections of the step fin. Physically speaking, the thinner the cross-section of the fin, the easier the heat can transfer from the fin material to the environment, and consequently the lower is the fin temperature. In fig. 6 we have plotted the effect of the thermal conductivity parameter on the temperature distribution within the fin. Results in the figure reveal that as the value of β increases the temperature distribution within both sections increases.

Figure 7 illustrates the effect of Biot number, *i. e.*, Bi, on the temperature distribution within the fin. Increasing the Biot number makes the convective cooling more effective. Therefore, as the Biot number increases, the cooling effects become more profound, which in turn causes the lowering of temperatures within both sections of the fin.

Conclusions

The performance analysis of convective step fin with temperature-dependent thermal conductivity is considered. The energy equation within each part, namely thin and thick parts, was considered separately. The two main nonlinear energy equations together with the joint boundary condition in the junction and other boundary conditions on base and tip of the fin build a system of ordinary differential equation. Differential transformation method (DTM) and variational iteration method (VIM) were used to analytically solve the non-linear system of equations. As the convection effect, *i. e*., Biot number, increases, this effect is to lower the fin temperature. Unlikely, as the thermal conductivity of the fin increases, *i. e*., the parameter β increases, it promotes slower cooling accompanied by higher local fin temperatures. As an important result, it was found that the DTM solution can achieve extremely accurate results when compared with the VIM. This paper proves that the DTM is a powerful analytical technique to handle non-linear energy equations in step fins.

Nomenclature

Bi – Biot number based on fin length
C_1 – constant which represents the temperature at the fin tip
C_2 – constant which represents the junction temperature
C_3 – constant which represent the junction temperature gradient for the thick section
h – heat transfer coefficient, [$Wm^{-2}K^{-1}$]
k – temperature-dependent thermal conductivity, [$Wm^{-1}K^{-1}$]
k_b – thermal conductivity at the base temperature, [$Wm^{-1}K^{-1}$]
L – length of the entire fin, [m]
ℓ – length of the thin section of the fin, [m]
T – temperature, [K]
T_b – fin's base temperature, [K]
T_1 – temperature within the thin section of the fin, [K]
T_2 – temperature within the thick section of the fin, [K]
T_∞ – ambient temperature, [K]
t – unreduced semi-thickness of the fin, [m]
x – axial co-ordinate for entire fin, [m]
x_1 – axial co-ordinate for the thin section of the fin, [m]
x_2 – axial co-ordinate for the thick section of the fin, [m]

Greek symbols

α – thickness ratio
β – thermal conductivity parameters
δ – dimensionless fin semi thickness
θ – dimensionless temperature within the thin section of the fin
κ – slope of the thermal conductivity--temperature curve, [K^{-1}]
λ – length ratio
ϕ – dimensionless temperature within the thick section of the fin
ξ – dimensionless axial co-ordinate of the thin section of the fin
ζ – dimensionless axial co-ordinate for the entire fin
τ – dimensionless axial co-ordinate of the thick section of the fin

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