EXACT SOLUTIONS OF TIME-FRACTIONAL HEAT CONDUCTION EQUATION BY THE FRACTIONAL COMPLEX TRANSFORM

by

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The Fractional Complex Transform is extended to solve exactly time-fractional differential equations with the modified Riemann-Liouville derivative. How to incorporate suitable boundary/initial conditions is also discussed.

Key words: *Fractional Complex Transform, Modified Riemann-Liouville Derivative, Time-fractional Heat Conduction Equation, Exact Solution*

1. Introduction

The fractional complex transform[1,2] was first proposed by Li and He in 2010 to convert the fractional differential equations with the modified Riemann-Liouville derivative [3-5] into ordinary differential equations, which can be easily solved using simple calculus.

2. Fractional Complex Transform

The fractional complex transform is to introduce a transform in the form[1,2]

$$\xi = \frac{qt^{\alpha}}{\Gamma(1+\alpha)} + \frac{px^{\beta}}{\Gamma(1+\beta)} + \frac{ky^{\gamma}}{\Gamma(1+\gamma)} + \frac{lz^{\lambda}}{\Gamma(1+\lambda)}. \tag{1}$$

where *p*, *q*, *k*, and *l* are unknown constants to be further determined, $0<\alpha\le 1$, $0<\beta\le 1$, $0<\gamma\le 1$, $0<\lambda\le 1$. When $\alpha=\beta=\gamma=\lambda=1$, the transform becomes the standard travelling wave transform $\xi = qt + px + ky + lz$. Consequently, the fractional complex transform is a natural extension of the travelling wave transform.

The purpose of introducing the fractional complex transform is to convert fractional differential equations with the modified Riemann-Liouville derivative[3-5] into ordinary differential equations, making the solution procedure much easy to access. For example, consider the following fractional differential equation:

$$\partial_t^{\alpha} u(x,t) = k\partial_x^{\beta} u(x,t),\ t\in R^{+},\ x\in R. \tag{2}$$

where k is a positive coefficient, $0<\alpha<1$, $0<\beta<1$, $u(x,t)$ is the real-valued variable, ∂_t^{α} and ∂_x^{β} are modified Riemann-Liouville derivatives [3-5].

Using the fractional complex transform, Eq. (1), we can convert Eq. (2) into a simple ordinary equation, which is

$$(kp-q)u_{\xi}=0 \tag{3}$$

It is extremely easy to solve the resultant equation.

3. Time-fractional Heat Conduction Equation

Now we consider the two-dimensional time-fractional homogeneous heat conduction equation of the form

$$\begin{cases} \dfrac{\partial^\alpha T}{\partial t^\alpha} = D(T_{xx} + T_{yy}), \\ t > 0, 0 < x < a,\ 0 < y < b, 0 < \alpha \le 1. \end{cases} \tag{4}$$

where the fractional derivative is the modified Riemann-Liouville derivative [3-5].

Using fractional complex transform

$$\xi = \frac{qt^\alpha}{\Gamma(1+\alpha)} + px + ky, \tag{5}$$

Eq. (4) becomes

$$D(p^2 + k^2)T_{\xi\xi} - qT_\xi = 0. \tag{6}$$

Solving Eq. (6) leads to the result

$$T(\xi) = c_1 + c_2 \exp\left\{ \frac{q\xi}{D(p^2 + k^2)} \right\}. \tag{7}$$

or

$$T(x, y, t) = c_1 + c_2 \exp\left\{ \frac{qpx}{D(p^2 + k^2)} + \frac{qky}{D(p^2 + k^2)} + \frac{q^2 t^\alpha}{D(p^2 + k^2)\Gamma(1+\alpha)} \right\}. \tag{8}$$

where c_1 and c_2 are integral constants.

4. Discussion

The fractional complex transform makes the solution procedure extremely simple. When contrasted with other analytical methods, such as the Heat-Balance Integral Method[6-8], the homotopy perturbation method[9,10], the variational iteration method[11], the exp-function method[12], and others[13,14], the present method combines the following two advantages: (1) the obtained ordinary equations are always very simple and can be easily solved; (2) the transform suggests a suitable way how to incorporate suitable boundary/initial conditions to the governing equations. It must be especially pointed out that the number of initial/ boundary conditions and the way they are imposed play a crucial role in fractional calculus , if inappropriate, it may lead to improper posedness of the problem or even nonexistence of the solution. The general initial condition for our problem can be expressed as

$$T(x, y, 0) = a + b\mathrm{e}^{cx+dy} \tag{9}$$

where *a,b,c,d* are constants. Consider an initial condition in the form

$$T(x, y, 0) = \exp\left\{ \frac{1}{3}x + \frac{2}{3}y \right\}. \tag{10}$$

From Eq. (8), we have

$$T(x,y,0)=c_1+c_2\exp\left\{\frac{q}{D(p^2+k^2)}(px+ky)\right\} \tag{11}$$

Comparing Eq.(10) with Eq.(11) implies that $c_1=0, c_2=1$, $p=1$, $k=2$ and $q=5D/3$, we, therefore, obtain the following exact solution:

$$T(x,y,t)=\exp\left\{\frac{1}{3}x+\frac{2}{3}y+\frac{5Dt^\alpha}{9\Gamma(1+\alpha)}\right\}. \tag{12}$$

In case the initial condition is not expressed in an exponential function, we can use the least squares technology to determine the constants in Eq.(8):

$$\int_0^a\int_0^b\left[c_1+c_2\exp\left\{\frac{q}{D(p^2+k^2)}(px+ky)\right\}-T(x,y,0)\right]^2 dxdy\to\min. \tag{13}$$

5. Conclusion

The fractional complex transform is easy to convert fractional differential equations to ordinary differential equations, and exact solutions can be easily obtained. The solution process is straightforward and concise. The paper can be used as a paradigm for other applications for fractional differential equations.

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