THE DIRIHLET PROBLEM FOR THE FRACTIONAL POISSON'S EQUATION WITH CAPUTO DERIVATIVES
A Finite Difference Approximation and a Numerical Solution

by

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A finite difference approximation for the Caputo fractional derivative of the $4-\beta$, $1 < b \leq 2$ order has been developed. A difference schemes for solving the Dirihlet's problem of the Poisson's equation with fractional derivatives has been applied and solved. Both the stability of difference problem in its right-side part and the convergence have been proved. A numerical example was developed by applying both the Liebman and the Monte-Carlo methods.

Key words: *fractional Caputo derivative, approximation, differential equations, numerical methods, stability*

Introduction

Fractional partial differential equations are generalizations of classical partial differential equations by replacing the integer-order derivatives by fractional-order derivatives [1, 2]. Because fractional derivatives provide useful tools for a description of memory and hereditary properties, the utility of fractional partial differential equations in mathematical modelling attracts more and more attention [3-5]. Different effective methods such as the fractional complex transform [6, 7], the homotopy perturbation method [8, 9], the variational iteration method [10], the exp-function method [11, 12], the heat-balance integral method [13-16], and others [17-19].

In this paper we consider the Poisson equation with fractional derivatives in the domain $D = \{0 < x < a,\ 0 < y < b\}$ concerning the solution $u(x, y)$ satisfying the equation:

$$ {}^{C}\mathrm{D}^{\beta}_{x+}u(x, y) + {}^{C}\mathrm{D}^{\beta}_{y+}u(x, y) = -f(x, y) \tag{1a} $$

with a boundary condition

$$ u\big|_{\Gamma} = \Psi(x, y) \tag{1b} $$

where Γ is the boundary of the area D, and the fractional order is $1 < \beta \leq 2$, and

$$ {}^{C}\mathrm{D}^{\beta}_{x+1}u(x, y) = \frac{1}{\Gamma(2-\beta)} \int\limits_{0}^{x} \frac{u''(s, y)}{(x-s)^{\beta-1}}\,\mathrm{d}s \tag{2a} $$

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$$ {}^{C}\mathrm{D}_{y+}^{\beta}u(x,y)=\frac{1}{\Gamma(2-\beta)}\int_{0}^{y}\frac{u''(x,s)}{(y-s)^{\beta-1}}\mathrm{d}s \tag{2b}$$

are fractional Caputo derivatives [1, 2] with respect the space variables x and y, respectively.

The time and space-fractional partial differential equation describe transport dynamics in complex systems governed by anomalous dispersion and non-exponential relaxation [3]. Because of complexity in the theoretic analysis of numerical approximation of fractional systems, the common approach is to apply the finite difference method to discretize fractional derivative operators, and then obtain the numerical solutions of the fractional partial differential equations. There exists significant interest in developing numerical methods for their solutions. Meerschaert *et al*. [17, 18] considered one dimensional Riemann-Liouville (R-L) fractional advection dispersion equation with variable coefficients on a finite domain [6] and two-sides space-fractional partial differential equations [18] by finite difference method. Muslih *et al*. [19] proposed a Fourier transform method to solve fractional Poisson's equation with Riesz fractional derivative. Tadjeran *et al*. [20] developed a second-order accurate numerical approximation for the fractional diffusional equation. Goloviznin *et al*. [21] developed a numerical method for solutions of some 1-D equations with fractional derivatives. Beibalaev [22, 23], developed a numerical method of the solution for heat transfer problems in media with fractal structures.

The outline of the paper is: (1) we consider a finite difference approximation of the Caputo derivative, and (2) Develop a numerical method and investigating the right-side stability and the convergence of the numerical scheme. (3) A numerical example by using the Liebman method [24] and the Monte Carlo approach based on the suggested finite difference approximation scheme is developed.

The numerical method

Finite difference approximation of the Caputo derivatives

From the definition of the Caputo derivative [2] in the interval $[x_{\mathrm{n}}, x_{n+1}]$ we obtain:

$$ ({}^{C}D_{x+}^{\beta}u)=\frac{1}{\Gamma(2-\beta)}\int_{x_{\mathrm{n}}}^{x_{n+1}}\frac{u''(s)}{(x_{x+1}-s)^{\beta-1}}\mathrm{d}s \tag{3}$$

where $\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x$ is the gamma function.

Expressing of the second order derivative with respect to the space variable $u''(x)$ by finite differences in the interval $[x_n, x_{n+1}]$ we get:

$$ \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)_n\approx\frac{u(x_{n+1})-2u(x_n)+u(x_{n-1})}{h^2} \tag{4a}$$

Hence, the finite difference approximation of the fractional derivative of order β is:

$$ ({}^{C}\mathrm{D}_{x+}^{\beta}u)_n\approx\frac{u(x_{n+1})-2u(x_n)+u(x_{n-1})}{\Gamma(3-\beta)h^{\beta}} \tag{4b}$$

Similarly, we have:

$$ ({}^{C}\mathrm{D}_{y+}^{\beta}u)_m\approx\frac{u(y_{m+1})-2u(y_m)+u(y_{m-1})}{\Gamma(3-\beta)l^{\beta}} \tag{5}$$

Substitution of the functions $u(x_n + h)$ and $u(x_n - h)$ developed as Taylor's series, in (4) yields:

$$\frac{u(x_n+h)-2u(x_n)+u(x_n-h)}{\Gamma(3-\beta)h^\beta} \approx \frac{h^2u''(x_n)+\dfrac{h^4}{12}u^{IV}(\xi_n)}{\Gamma(3-\beta)h^\beta} \approx ({}^C\mathrm{D}^\beta_{x+}u)_n + ah^{4-\beta} \tag{6}$$

where $a = M/12\Gamma(3-\beta)$, $M = \max|u^{\mathrm{IV}}(x_n)|$.

The numerical method

For finding the solution of the problem (1) in the area $\overline{D} = \{0 \le x \le a, 0 \le y \le b\}$ let us introduce a grid $\Omega = \{z_{n,m} = (x_n, y_m), n = 0, 1, \dots N, m = 0,1, \dots M\}$ where $x_n = nh$, $y_m = ml$, $h = a/N$, $l = b/M$. Then, using the equalities expressions (4b) and (5) we get the difference scheme:

$$\frac{u(x_{n+1},y_m)-2u(x_n,y_m)+u(x_{n-1},y_m)}{\Gamma(3-\beta)h^\beta} + \\ + \frac{u(x_n,y_{m+1})-2u(x_n,y_m)+u(x_n,y_{m-1})}{\Gamma(3-\beta)l^\beta} = -f(x_n,y_m) \tag{7}$$

$$(x_n, y_m) \in \omega,$$

$$u(x_n, y_m) = \psi(x_n, y_m),\ (x_n, y_m) \in \gamma \tag{8}$$

where $\omega = \{z_{n,m} = (x_n, y_m), n = 0,1, \dots, N-1, m = 0, 1, \dots, M-1\}$ is a set of internal nodes of the grid Ω, $\gamma = \{z_{0,m}, z_{N,m}\}_{m-1}^{M} \cup \{z_{n,0}, z_{n,\mathrm{M}}\}_{n=1}^{N}$ is a set of boundary nodes.

For seek convenience eq. (7) can be expressed in a form with respect to $u_{n,m}$, *i. e.* in a form:

$$\left(\frac{2}{\Gamma(3-\beta)h^\beta} + \frac{2}{\Gamma(3-\beta)l^\beta}\right)u_{n,m} = \frac{u_{n+1,m}+u_{n-1,m}}{\Gamma(3-\beta)h^\beta} + \frac{u_{n,m+1}+u_{n,m-1}}{\Gamma(3-\beta)l^\beta} + f_{n,m} \tag{9}$$

Let us denote as z, *i. e.* $z_{n,m} = (x_n, y_m)$ the central point of a template, approximating eq. (1a) . Moreover, $Z(x)$ denotes the entire template of five points $z_{n,m}, z_{n\pm1,m}, z_{n,m\pm1}$, while denotes $Z'(x)$ all points of $Z(x)$ except the central one $z_{n,m} = (x_n, y_m)$, *i. e.* $Z'(x)$ is a template of four points $z_{n\pm1,m}, z_{n,m\pm1}$. Then, eq. (9) may be expressed as:

$$A(z)\,u(z) = \textstyle\sum_{\xi\in Z'(z)} B(z,\xi)\,u(\xi) + F(\xi) \tag{10a}$$

$$A(z) = \frac{2}{\Gamma(3-\beta)}\left(\frac{1}{h^\beta} + \frac{1}{l^\beta}\right) \tag{10b, c}$$

$$B(z, z_{n\pm1,m}) = \frac{1}{\Gamma(3-\beta)h^\beta}, \quad B(z, z_{n,m\pm1}) = \frac{1}{\Gamma(3-\beta)l^\beta} \tag{10c, d}$$

$$A(z) > 0,\ B(z,\xi) > 0,\ F(z) = f(x_n, y_m) \tag{10e}$$

Denoting $L^\beta(z)u(z) = A(z)u(z) - \sum_{\xi\in Z'(z)} B(z,\xi)u(\xi)$ we write down eq. (9) as:

$$L^\beta(z)u(z) = F(z),\quad z \in \omega,\quad u(z) = \psi(z),\quad z \in \gamma \tag{11}$$

Let us express the solution (11) as a sum $u(z)=\tilde{u}(z)+\bar{u}(z)$, where $\bar{u}(z)$ is the solution of a homogeneous equation with non-homogeneous boundary conditions:

$$L^{\beta}\,\tilde{u}(z)=0,\quad z\in\omega,\quad \tilde{u}(z)=\psi(z),\quad z\in\gamma \tag{12}$$

and $\bar{u}(z)$ is the solution of non-inhomogeneous equation with a homogeneous boundary condition, *i. e.:*

$$L^{\beta}\bar{u}(z)=F(z),\quad z\in\omega,\ \bar{u}(z)=0,\quad z\in\gamma \tag{13}$$

With respect to solution (11), all the conditions of the maximum principle are satisfied, and we have:

$$\|\tilde{u}\|_{C(\Omega)}\le\|\psi\|_{C(\gamma)},\quad \|u\|_{C(\Omega)}=\max_{z\in\Omega}|u(z)|,\quad \|\psi\|_{C(\gamma)}=\max_{z\in\gamma}|\psi(z)| \tag{14a,b,c}$$

After simple transformations we read

$$u_{n,m}=\frac{1}{2(h^{\beta}+l^{\beta})}[l^{\beta}(u_{n+1,m}-u_{n-1,m})+h^{\beta}(u_{n,m+1}+u_{n,m-1})+\Gamma(3-\beta)h^{\beta}l^{\beta}f_{n,m}] \tag{15}$$

Right-side stability and convergence

Stability

Let us construct a majorant function for solving the problem (13) and apply the comparison theorem, namely:

$$Y(z)=K(a^2+b^2-x^{\beta}-y^{\beta}) \tag{16}$$

where K is an arbitrary constant, while a and b are the length of the sides of the rectangle D. It is obvious that $Y(z)\ge 0$ for all $z\in\Omega$. Then, let us denote $D(z)=A(z)-\sum_{\xi\in Z'(z)}B(z,\xi)$ and calculate the expression:

$$LY(z)=D(z)Y(z)+\sum_{\xi\in Z'(z)}B(z,\xi)[Y(z)-Y(\xi)] \tag{17}$$

using the function (16) for all $z\in\omega$. Then, the function $LY(z)$ can be expressed as:

$$LY(z)={}^{C}\mathrm{D}^{\beta}_{x+}Y+{}^{C}\mathrm{D}^{\beta}_{y+}Y \tag{18}$$

According to the definition of the fractional derivation in Caputo sense for $1<\beta\le 2$ we have:

$$\begin{aligned}
{}^{C}\mathrm{D}^{\beta}_{x+}Y&=\frac{K}{\Gamma(2-\beta)}\int_0^x\frac{(s^{\beta})''}{(x-s)^{\beta-1}}\mathrm{d}s=\frac{K\beta(\beta-1)}{\Gamma(2-\beta)}\int_0^x s^{\beta-2}(x-s)^{1-\beta}\mathrm{d}s=\begin{Bmatrix}s=\tau x\\ \mathrm{d}s=x\mathrm{d}\tau\end{Bmatrix}=\\
&=\frac{K\beta(\beta-1)}{\Gamma(2-\beta)}\int_0^1 x^{\beta-2}\tau^{\beta-2}x^{1-\beta}(1-\tau)^{1-\beta}x\mathrm{d}\tau=\frac{K\beta(\beta-1)}{\Gamma(2-\beta)}\int_0^1\tau^{\beta-2}(1-\tau)^{1-\beta}\mathrm{d}\tau=\\
&=\frac{K\beta(\beta-1)}{\Gamma(2-\beta)}B(\beta-1,2-\beta)=\frac{K\beta(\beta-1)\Gamma(\beta-1)\Gamma(2-\beta)}{\Gamma(2-\beta)\Gamma(1)}=K\Gamma(\beta+1)
\end{aligned} \tag{19}$$

where $B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t$ is the Euler's beta function.

Similarly, with ${}^{C}\mathrm{D}_{y+}^{\beta}Y = K\Gamma(\beta+1)$ we have $LY(z) = 2K\Gamma(\beta+1)$ and it may be considered that $Y(z)$ is a solution of the boundary problem:

$$LY(z) = \overline{F}(z), \quad z \in \omega \tag{20a}$$

$$Y(z) = \overline{\psi}(z), \quad z \in \gamma \tag{20b}$$

where $\overline{F}(z) = LY(z) = 2K\Gamma(\beta+1)$ and $\overline{\psi}(z) \geq 0$ are the values of function (16) for $z \in \gamma$.

If $K = [1/2\Gamma(\beta+1)]\|F\|_{C(\omega)}$ then all conditions of the comparison theorem [24] concerning the problems (12) and (13) are satisfied. From the comparison theorem it follows that:

$$\|\overline{u}\|_{C(\Omega)} \leq \max_{z\in\Omega}.Y(z) \leq K(a^2+b^2) \quad \text{and} \quad \|\overline{u}\|_{C(\Omega)} \leq \frac{a^2+b^2}{2\Gamma(\beta+1)}\|F\|_{C(\omega)} \tag{21a,b}$$

Notice that eq. (21b) follows from the choice of K defined above. Moreover, according to the triangle inequality and (14a,bc), concerning the solutions of (7) and (8) we get the following estimations about the stability of the difference scheme of the right-side part f and the boundary conditions Ψ, namely

$$\|z\|_{C(\Omega)} \leq \frac{a^2+b^2}{2\Gamma(\beta+1)}\|f\|_{C(\omega)} + \|\psi\|_{C(\gamma)} \tag{22}$$

The constants in (22) are independent of the grid steps h and l, and the difference scheme (7) is stable.

Convergence and error estimation

Let us denote $\eta_{n,m} = z_{n,m} - u(x_n, y_m)$. Here $Z_{n,m}$ is a solution of the difference problem (7), and $u(x, y)$ is a solution of the (1a). Then, substituting $z_{n,m} = \eta_{n,m} + u(x_n, y_m)$ in eq. (7) we see that the error satisfies the equation:

$$\frac{\eta_{n+1,m} - 2\eta_{n,m} + \eta_{n-1,m}}{\Gamma(3-\beta)h^{\beta}} + \frac{\eta_{n,m+1} - 2\eta_{n,m} + \eta_{n,m-1}}{\Gamma(3-\beta)l^{\beta}} = -\psi_{n,m} \tag{23a}$$

$$\psi_{n,m} = \frac{u_{n+1,m} - 2u_{n,m} + u_{n-1,m}}{\Gamma(3-\beta)h^{\beta}} + \frac{u_{n,m+1} - 2u_{n,m} + u_{n,m-1}}{\Gamma(3-\beta)l^{\beta}} + \Gamma(3-\beta)h^{\beta}l^{\beta}f_{n,m} \tag{23b}$$

$$(x_n, y_m) \in \omega, \quad \eta_{n,m} = 0, \quad (x_n, y_m) \in \gamma \tag{23c, d}$$

Let us impose some constraints to the function $u(x, y)$. Firstly, let the fourth order derivative of the solution $u(x, y)$ are constrained. Then the approximation error has an order of magnitude $4-\beta$.Thus, there exists a constant M_1 with is independent of h and l, so that $\|\psi\|_{C(\Omega)} \leq$ $\leq M_1(h^{4-\beta} + l^{4-\beta})$. Notice that the problem (23) differs from the difference scheme (7) only with respect to the right-side parts. Therefore, the following estimation is correct, namely:

$$\|\eta\|_{C(\Omega)} \leq \frac{a^2+b^2}{2\Gamma(\beta+1)}\|\psi\|_{C(\omega)} \Rightarrow \|\eta\|_{C(\Omega)} \leq M_2(h^{4-\beta} + l^{4-\beta}) \tag{24a, b}$$

where $M_2 = [1/\Gamma(\beta+1)]M_1(a^2+b^2)$ is a constant independent of h and l. Therefore, the difference scheme (8) is convergent and has an order of approximation $h^{4-\beta} + l^{4-\beta}$.

The Monte-Carlo method to the set of difference equations developed

Solution scheme

Let us consider a case when $h = l$ that transforms the set (7) as:

$$u_{n,m} = \frac{1}{4}[(u_{n+1,m} + u_{n-1,m}) + (u_{n,m+1} + u_{n,m-1}) + \Gamma(3-\beta)h^{\beta} f_{n,m}] \tag{25}$$

where $u_{n,m}$ is the approximated value of $u(nh, mh)$ for $1 \le n$ and $m \le L-1$.

If the point (nh, mh) is at the boundary, then the value of $u_{n,m}$ is known and equals $\psi(nh, mh), f_{n,m} = f(nh, mh)$. The relationship (25) is, in fact, a "complete mathematical expectation" if to assume $f_{n,m} = 0$. Hence, the average values $\xi_{n,m}$ meet the following:

(1) A point matching the node (nh, mh), suggests the initial value of the counter to be $[\Gamma(3-\beta)h^{\beta}f_{n,m}]/4$;

(2) With almost equal probabilities we can move the point to one of neighboring nodes, by adding to the counter the proper value of $[\Gamma(3-\beta)h^{\beta}f_{n,m}]/4$.

Then execute (2) again, and move to the next node till reaching the boundary. After reaching the boundary the appropriate value of ψ is added to counter, and "trajectory" stops.

The effective value of the counter provides a selective casual value $\xi_{n,m}$. It is obvious that the values $\xi_{n,m}$ satisfy the set (25), *i. e.* $M\xi_{n,m} = u_{n,m}$. Thus, we get, un,m $\approx \sum_{i=1}^{N} \xi_i / N$, where is the number of "trajectories" (ξ_i is the result of i^{th} trajectory), simulated from the initial point (nh, mh).

As an example we consider a problem solved by the Liebman and the Monte Carlo methods.

Example problem: a numerical solution

Find the solution of:

$${}^{C}\mathrm{D}_{x+}^{\beta} u(x, y) + {}^{C}\mathrm{D}_{y+}^{\beta} u(x, y) = -10[4 + \pi^2 (x - 2x^2)]\sin \pi y \tag{26}$$

defined in $\mathrm{D} = \{0 < x < 0.5, 0 < y < 1\}$ and obeying the boundary condition $u_{\Gamma} = 0$.

Two methods, those of Liebman [24] and the Monte-Carlo method were used. Numerical data are summarized in tabs. (1-4) , while graphical example are illustrated by figs. (1-4) for two fractional orders for $\beta = 2$ and $\beta = 1.5$.

In the figures there are the diagrams of the problem (26) solution for $\beta = 2$ and $\beta = 1.5$. In all cases illustrated the increase in the fractional order β reduces the magnitudes of the peaks.

Table 1. Solution of the problem $\beta = 2$ by the Liebman's method (numerical data)

y \ x	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0	0	0	0	0	0	0	0	0	0	0	0
0.1	0	0.139	0.265	0.365	0.429	0.451	0.429	0.365	0.265	0.139	0
0.2	0	0.248	0.471	0.648	0.762	0.801	0.762	0.648	0.471	0.248	0

Table 1. Continuation

y \ x	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0.3	0	0.325	0.618	0.851	1.001	1.052	1.001	0.851	0.618	0.325	0
0.4	0	0.371	0.707	0.972	1.143	1.202	1.143	0.972	0.707	0.371	0
0.5	0	0.387	0.736	1.013	1.191	1.253	1.191	1.013	0.736	0.387	0
0.6	0	0.371	0.707	0.972	1.143	1.202	1.143	0.972	0.707	0.371	0
0.7	0	0.325	0.618	0.851	1.001	1.052	1.001	0.851	0.618	0.325	0
0.8	0	0.248	0.471	0.648	0.762	0.801	0.762	0.648	0.471	0.248	0
0.9	0	0.139	0.265	0.365	0.429	0.451	0.429	0.365	0.265	0.139	0
1	0	0	0	0	0	0	0	0	0	0	0

Table 2. Solution of the problem for β = 2 by the Monte Carlo method (numerical data)

y \ x	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0	0	0	0	0	0	0	0	0	0	0	0
0.1	0	0.153	0.249	0.355	0.374	0.425	0.410	0.333	0.268	0.136	0
0.2	0	0.251	0.485	0.609	0.682	0.724	0.692	0.639	0.423	0.277	0
0.3	0	0.347	0.672	0.831	0.925	0.997	0.973	0.851	0.674	0.347	0
0.4	0	0.419	0.735	0.940	1.114	1.185	1.126	1.015	0.752	0.442	0
0.5	0	0.436	0.800	1.027	1.212	1.262	1.142	1.037	0.815	0.451	0
0.6	0	0.426	0.749	0.964	1.153	1.222	1.181	1.014	0.769	0.431	0
0.7	0	0.331	0.644	0.849	0.938	1.021	0.980	0.838	0.643	0.370	0
0.8	0	0.251	0.463	0.618	0.739	0.744	0.673	0.611	0.516	0.262	0
0.9	0	0.141	0.243	0.325	0.385	0.390	0.377	0.322	0.241	0.117	0
1	0	0	0	0	0	0	0	0	0	0	0

Table 3. Solution of the problem for β = 1.5 by the Liebman's method (numerical data)

y \ x	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0	0	0	0	0	0	0	0	0	0	0	0
0.1	0	1.124	2.176	2.855	3.262	3.397	3.262	2.855	2.176	1.124	0
0.2	0	2.329	4.138	5.430	6.204	6.462	6.204	5.430	4.138	2.329	0
0.3	0	3.205	5.696	7.473	8.539	8.895	8.539	7.473	5.696	3.205	0
0.4	0	3.768	6.696	7.473	8.539	8.895	8.539	7.473	6.696	3.768	0
0.5	0	3.962	7.041	9.238	10.555	10.994	10.555	9.238	7.041	3.962	0
0.6	0	3.768	6.696	7.473	8.539	8.895	8.539	7.473	6.696	3.768	0
0.7	0	3.205	5.696	7.473	8.539	8.895	8.539	7.473	5.696	3.205	0
0.8	0	2.329	4.138	5.430	6.204	6.462	6.204	5.430	4.138	2.329	0
0.9	0	1.124	2.176	2.855	3.262	3.397	3.262	2.855	2.176	1.124	0
1	0	0	0	0	0	0	0	0	0	0	0

Table 4. Solution of the problem for β = 1.5 by the Monte Carlo method (numerical data)

y \ x	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0	0	0	0	0	0	0	0	0	0	0	0
0.1	0	1.280	1.971	2.886	3.179	3.344	3.231	2.824	2.331	1.341	0
0.2	0	2.349	4.108	5.308	6.484	6.866	6.277	5.513	4.173	2.193	0
0.3	0	3.408	5.896	7.246	8.118	8.887	8.651	7.596	5.755	3.044	0
0.4	0	3.760	6.857	8.812	10.513	10.516	10.443	8.520	7.356	3.742	0
0.5	0	3.942	7.455	9.045	10.711	11.436	10.834	9.409	6.674	3.945	0
0.6	0	3.653	7.304	9.007	9.685	10.668	9.956	8.560	6.891	3.566	0
0.7	0	3.022	5.956	7.411	8.776	8.944	8.870	7.417	6.047	3.485	0
0.8	0	2.187	4.208	5.296	6.116	6.784	6.329	5.334	4.061	2.148	0
0.9	0	1.289	2.394	2.692	3.164	3.357	3.309	2.799	1.984	1.311	0
1	0	0	0	0	0	0	0	0	0	0	0

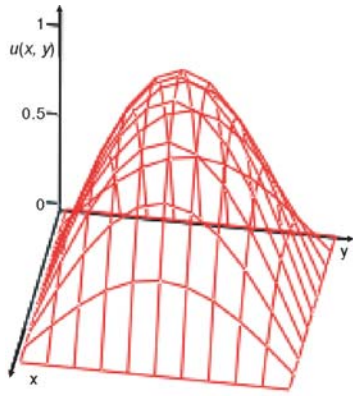

Figure 1. Solution of the problem (26) by the Liebman's method for β = 2

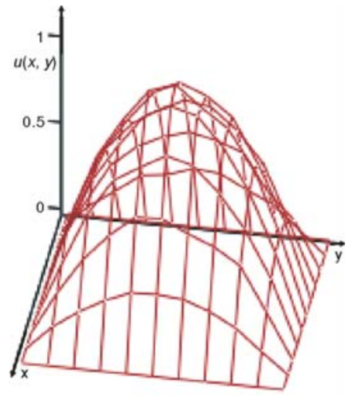

Figure 2. Solution of the problem (26) by the Monte Carlo method for β = 2

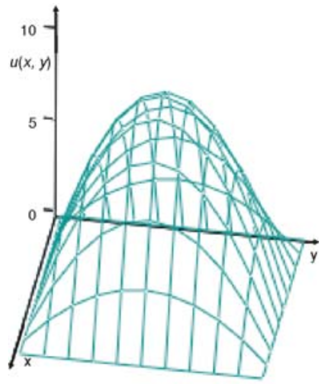

Figure 3. Solution of the problem (26) by the Liebman's method for β = 1.5

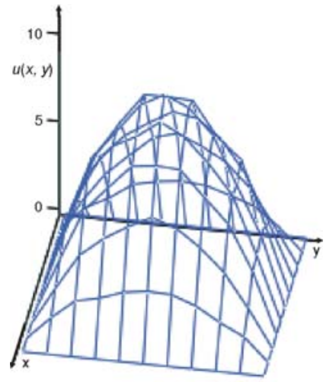

Figure 4. Solution of the problem (26) by the Monte Carlo method for β = 1.5

Conclusions

Because of complexity in the theoretic analysis of numerical approximation of fractional systems, the common approach is to apply the finite difference method to discretize fractional derivative operators, then obtain the numerical solutions of the fractional partial differential equations. In this paper we discussed a finite difference approximation of time-space-fractional PDE (1) with Caputo fractional derivatives. The stability and the convergence of the right-side of the approximated problem were proved. Computations by the suggested finite difference scheme were performed allowing applying both the Liebman and the Monte Carlo methods for numerical solutions.

References

[1] Samko, S. G., Kilbas, A. A., Marichev, O. I., Fractional Integrals and Derivatives: Theory and Applications (translation from Russian), Gordon and Breach, Amsterdam, 1993
[2] Nakhushev, A. M., Elements of Fractional Calculation and Their Application (in Russian), Nalchik, Russia, 2003
[3] Metzler, R., Klafter, J., The Random Walk's Guide to Anomalous Diffusion: A Fractional Dynamics Approach, *Phys. Rep., 339* (2000), 1, pp. 1-77
[4] Chen, C., Liu, F., Burrage, K., Finite Difference Methods and a Fourier Analysis for the Fractional Reaction-Subdiffusion Equation, *Applied Mathematics and Computation, 198* (2008), 2, pp. 754-769
[5] Yuste, S. B., Acedo, L., Lindenberg, K., Reaction Front in an A + B ?C Reaction-Subdiffusion Process, *Phys. Rev. E , 69* (2004), 3, 036126
[6] Li, Z. B., He, J.-H., Fractional Complex Transform for Fractional Differential Equations, *Mathematical and Computational Applications, 15* (2010), 5, pp. 970-973
[7] Li, Z. B., An Extended Fractional Complex Transform, *Journal of Nonlinear Science and Numerical Simulation, 11* (2010), Suppl., pp. 0335-0337
[8] Jafari, H., *et al*., Homotopy Perturbation Pade Technique for Solving Fractional Riccati Differential Equations, *Int. J. Nonlinear Sci. Num., 11* (2010), 2, pp. 271-275
[9] Golbabai, A., Sayevand, K., The Homotopy Perturbation Method for Multi-Order Time Fractional Differential Equations, *Nonlinear Science Letters A, 1* (2010), 2, pp. 147-154
[10] He, J.-H., Approximate Analytical Solution for Seepage Flow with Fractional Derivatives in Porous Media, *Computer Methods in Applied Mechanics and Engineering, 167* (1998), 1-2, pp. 57-68
[11] Tadjeran, C., Meerschaert, M. M., Scheeffler, H.-P., A Second-Order Accurate Numerical Approximation for the Fractional Diffusion Equation, *Journal of Computational Physics, 213* (2006), 1, pp. 205-213
[12] Zhang, S., Zhang, H. Q., Fractional Sub-Equation Method and Its Applications to Nonlinear Fractional PDEs, *Physics Letters A, 375* (2011), 7, pp. 1069-1073
[13] Hristov, J., Approximate Solutions to Fractional Subdiffusion Equations, *European Physical Journal, 193* (2011), 1, pp. 229-243
[14] Hristov, J., Starting Radial Subdiffusion from a Central Point through a Diverging Medium (A Sphere): Heat-Balance Integral Method, *Thermal Science, 15* (2011), Suppl. 1, pp. S5-S20
[15] Hristov, J., Heat-Balance Integral to Fractional (Half-Time) Heat Diffusion Sub-Model, *Thermal Science, 14* (2010), 2, pp. 291-316
[16] Hristov, J., A Short-Distance Integral-Balance Solution to a Strong Subdiffusion Equation: A Weak Power-Law Profile, *Int. Rev.Chem. Eng., 2* (2010), 5, pp. 555-563
[17] Meerschaert, M. M., Tadjeran, C., Finite Difference Approximations for Fractional Advection-Dispersion Flow Equations, *J. Comput. Appl. Math., 172* (2004), 1, pp. 65-77
[18] Meerschaert, M. M., Tadjeran, C., Finite Difference Approximations for Two-Sides Space-Fractional Partial Differential Equations, *Appl. Num. Math., 56* (2006), 1, pp. 80-90
[19] Muslih,S. I., Agrawal, O. P., Riesz Fractional Derivatives and Fractional Dimensional Space, *Int J Theor Phys , 49* (2010), 2, pp. 270-275
[20] Tadjeran, C., Meerschaert, M. M., Scheffler, H.-P., A Second-Order Accurate Numerical Approximation for the Fractional Diffusional Equation, *J Comput Phy., 213* (2006), 1, pp. 205-213

[21] Goloviznin, V. M., Korotkin, I. A., Methods of the Numerical Solutions Some One-Dimensional Equations with Fractional Derivatives (in Russian), *Differential Equations, 42* (2006), 7, pp. 21-130
[22] Beibalaev,V. D., A Numerical Method of the Mathematical Model Solution for Heat Transfer in Media with Fractal Structure (in Russian), *Fundamental Research* (Moscow), *5* (2007), 12, pp. 249-251
[23] Beibalaev, V. D., A Mathematical Model of Transfer in Mediums with Fractal Structure (in Russian), *Math. Model.* (Moscow) *21* (2009), 5, pp. 55-62
[24] Samarsky, A. A., Gulin, A. V., Numerical Methods (in Russian), Nauka, Moscow, 1989

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