APPLICATION OF FRACTIONAL CALCULUS IN GROUND HEAT FLUX ESTIMATION

by

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Ground (soil) heat flux is important physical factor primarily because of its role in surface energy balance, analysis of atmospheric boundary layer, and land surface-atmosphere interaction. Direct measurement of this property is often associated with difficulties arising from need for adequate calibration of measuring devices, determination of proper depth for probes, upward water migration, and accumulation below measuring plates to lack of understanding of the governing thermal processes occurring at the ground surface. In the following paper approach for inferring heat flux indirectly, from known ground surface temperature time-dependant functions, using previously developed fractional diffusion equation for ground heat conduction is elaborated. Fractional equation is solved for two, most frequently encountered harmonic surface temperature functions. Yielded results were compared with analytic solutions. Validation results indicate that solutions obtained with fractional approach closely correspond to analytic solutions with remark that former are more general, containing the term covering the transitional effect.

Key words: *ground heat flux, fractional calculus, fractional diffusion equation*

Introduction

Climatic conditions on the Earth's surface are in part a function of varying physical position (elevation and latitude) and the influence of large-scale meteorological forces such as air and ocean currents [1]. Earth (soil/ground) surface can be defined here as an active layer for radiation, heat and mass exchange between the ground beneath that layer, and atmosphere below it. It has profound effect on micro climate in ground proximity. Properties of the ground surface affect the ratio of absorbed and reflected sunlight energy and accordingly energy balance of ground surface.

The energy balance for the ground surface can be presented schematically as in fig. 1 [2]. Dominant part of incident solar radiation is used for evaporation of water contained in ground, partly reflects back to atmosphere in form of shortwave and longwave terrestrial radiation and dissipates as sensible heat transfer to the atmosphere. Just a small part of incoming en-

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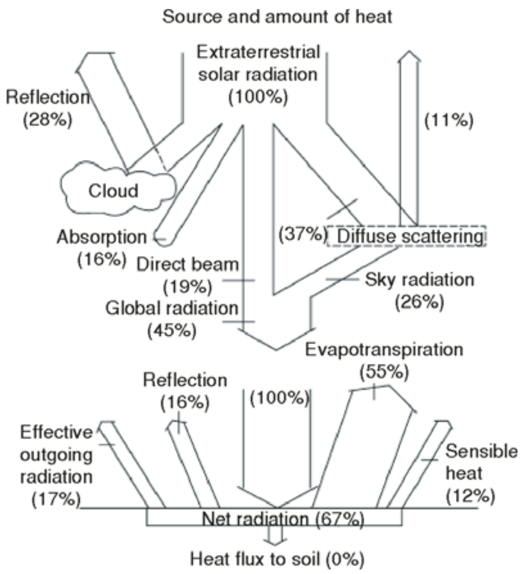

Figure 1. Daytime heat balances (adopted from [2], after Baver *et al*. [3])

ergy penetrates into the soil. This fraction of energy is referred to as ground (soil) heat flux or heat flux density.

At the beginnings of soil physics science, the process of heat transfer in soil was not properly understood. According to Sauer [4], Patten [5] can be credited as being the pioneer in providing the quantitative treatment of heat transfer processes in ground by measuring the thermal properties of several soils under controlled laboratory conditions.

Today, following equation is most frequently used for describing energy balance at the ground surface:

$$R_n - G = H - LE \tag{1}$$

where R_n [Wm^{-2}] is the net radiation – the net difference between incoming shortwave and longwave (terrestrial) radiation, G [Wm^{-2}] – the ground heat flux – a measure of the amount of energy moving into or out of the soil, which determines ground temperature and the rate of daily and seasonal temperature change, H [Wm^{-2}] – the sensible heat flux – energy involved in heating or cooling the air layer near the ground surface and LE [Wm^{-2}] – the latent heat flux – the amount of energy consumed by evaporating water or released during dew formation [1, 4, 6]. The equation stands if heating sources (from earth crust *e. g.* inner geothermal energy sources) are not present. All of the terms in equation depend on ground surface temperature.

In eq. (1), ground heat flux is the smallest term. That was the reason why it had been previously omitted from heat balance equation (set to zero), parameterized from meteorological parameters (as a fixed percentage of the net radiation) or measured with simple techniques (by using the output of heat flux plate without compensation) [7]. However, significant errors with this approach (especially for bare dry soils, during night time and morning) noticed lately, have indicated importance of this term.

Ground heat flux has profound effect on ground temperature profile which consequently affects microclimatic conditions on observed site and consequently rate of biological and chemical processes of plants. According to Sauer *et al*. [1], ground heat flux can be measured by one of the following methods: flux plate, calorimetric, gradient, or combination. Advantages as well as disadvantages of aforementioned methods are extensively covered in literature [8-12]. However, precise heat flux measurement is often accompanied with various problems arising from the lack of understanding of processes occurring at the ground surface, to difficulties with determining proper depth for probes (plates) placement as well as problems with coupling of energy and water transfer affecting the accuracy of obtained data [7, 13, 14]. In order to avoid aforementioned problems, Wang *et al*. [15] proposed method for inferring ground heat flux indirectly, from available time series of ground surface temperature using the fractional calculus. The method was tested using the data generated numerically and with data from two field experiments. Authors obtained satisfactory results with proposed method. Our primary aim in this article is to review that method and inter alia:

- provide analytic solution for thermal regime (pattern of ground temperature fluctuations by depth) at any time (covering transient regime) for arbitrary as well as for most common ground surface temperature time functions,
- review procedure for development of fractional ground heat conduction equation following the procedure proposed by Kulish [16] and particularly Agrawal [17] for arbitrary ground surface temperature time dependent functions and solve fractional equation for particular surface temperature time dependent function, and
- compare analytic and fractional heat diffusion equation solutions and provide suggestion where method (approach) could be successfully applied.

The analytic solution of ground heat conduction equation

Heat propagation in ground can be regarded as special case of diffusion equation. According to Fourier law, heat flux (for homogenous body) is proportional to temperature gradient:

$$q = -\lambda \nabla T \tag{2}$$

where q [Wm^{-2}] is the heat flux, λ [Wm^{-1}K^{-1}] – the thermal conductivity, and ∇T – the spatial gradient of temperature. In 1-D form (for z co-ordinate), law gets the following form:

$$q = -\lambda \frac{\partial T}{\partial z} \tag{3}$$

where $\partial T/\partial z$ is the temperature gradient in vertical direction representing ground depth. In this case, assumption is that ground is isotropic. Equation (3) is sufficient to describe heat conduction under the steady-state, but insufficient to cover non-steady (transient) conditions. In order to include this effect, 1-D heat diffusion equation for heat transport in ground needs to be invoked:

$$\rho c \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial z} \tag{4}$$

where t [s] is the time, ρ[kgm^{-3}] – the soil density, c [J kg^{-1}K^{-1}] – the soil specific heat capacity per unit mass, while product ρc [Jm^{-3}K^{-1}] represents the specific heat capacity per unit volume. Combining eq. (3) and (4), leads to the second law of heat conduction:

$$\rho c \frac{\partial T}{\partial t} = -\frac{\partial}{\partial z}\left(-\lambda \frac{\partial T}{\partial z}\right) \tag{5}$$

or, provided that the coefficient of soil thermal conductivity does not vary significantly with depth:

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial z^2} \tag{6}$$

where a [m^{2}s^{-1}] is the coefficient of thermal diffusivity.

Starting with the assumption that the ground properties are uniform over the entire observed depth, and taking into account initial:

$$T(z, 0) = T_0, \quad 0 \le z \le \infty \tag{7}$$

as well as boundary conditions:

$$T(0, t) = \mathrm{f}(\tau), \quad t > 0 \tag{8}$$

$$T(\infty, t) = 0, \quad t > 0 \tag{9}$$

the overall form of solution for ground temperature profile (thermal regime) can be derived from formula for temperature distribution in half-space when surface temperature is time-dependent using the Duhamel theorem [18], Fourier transformation [19] Laplace transformation [19] or combination of Laplace and Fourier transformations [19]. Solution has the following form:

$$T(z,t) = \frac{z}{\sqrt{4\pi a}} \int_0^t \frac{\mathrm{f}(\tau)}{\sqrt{(t-\tau)^3}} \exp\left[-\frac{z^2}{4a(t-\tau)}\right] \mathrm{d}\tau \tag{10}$$

which, after putting substitution:

$$\eta = \frac{z}{\sqrt{4a(t-\tau)}} \tag{11}$$

becomes:

$$T(z,t) = \frac{2}{\sqrt{\pi}} \int_{z/\sqrt{4at}}^{\infty} \exp(-\eta^2) \mathrm{f}\left(t - \frac{z^2}{4a\eta^2}\right) \mathrm{d}\eta \tag{12}$$

Only in this form solution satisfies the initial and boundary conditions. Alternatively, eq. (10) can be written in the following form [20]:

$$T(z,t) = \int_0^t \frac{\mathrm{d}\mathrm{f}(\tau)}{\mathrm{d}\tau} \mathrm{erfc}\left[-\frac{z}{\sqrt{4a(t-\tau)}}\right] \mathrm{d}\tau \tag{13}$$

In equations (7)-(13) T_0 is the initial temperature of the surface, $\mathrm{f}(\tau)$ – the arbitrary surface temperature time-dependent function, and erfc – the error complementary function.

As to the ground heat flux, it can be derived from eq. (3), making use of eq. (13) and taking into account that:

$$\mathrm{erfc}(z) = 1 - \mathrm{erf}(z) \tag{14}$$

as well as:

$$\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{erf}z) = -\frac{2}{\sqrt{\pi}} \mathrm{e}^{-z^2} \tag{15}$$

After computations, ground heat flux obtains the following form:

$$q(z,t) = \sqrt{\frac{\lambda c \rho}{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} \exp\left(-\frac{z^2}{4a(t-\tau)}\right) \frac{\mathrm{d}\mathrm{f}(\tau)}{\mathrm{d}\tau} \mathrm{d}\tau \tag{16}$$

For ground surface (*i. e.* setting $z = 0$), eq. (16) simplifies to:

$$q(0,t) = \sqrt{\frac{\lambda c \rho}{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau} \mathrm{d}\tau \tag{17}$$

In order to solve the integrals in eqs. (10), (13), (16), and (17) surface temperature time function f(τ) needs to be defined. Ground surface temperature continuously fluctuates as a response to changing meteorological conditions. These fluctuations are consequences of regular, periodic influences (successions of days and nights, and of summer and winter) and irregular episodic effects (cloudiness, cold or warm spells, rain or snowstorms, and periods of drought) [21]. Moreover, ground surface temperature is affected by ground thermal properties such as heat capacity and thermal conductivity as well as geographic location and canopy/vegetation cover. The most straightforward way to model the surface temperature fluctuations is to estimate it as harmonic function of time around the average value. Regardless the fact that this approach does not provide satisfactory results in certain cases [22, 23], it is still most commonly used. Hence, analytic solutions of integrals will be provided for these most frequently encountered cases.

Case 1 – Cosine ground surface temperature model

One of the first attempts to model ground surface temperature fluctuations was to describe it with cosine function [24]:

$$\mathrm{f}(\tau) = T(0, t) = \Delta T \cos(\omega t) \tag{18}$$

where ΔT is the amplitude of the surface temperature fluctuation, $\omega = 2\pi/t_0$ (t_0 – the radial period) – the radial frequency. The authors have demonstrated that, owing to Duhamel's theorem, analytic solution for ground temperature profile could be obtained in the following form (procedure is explained step-by-step in [18], pp. 205-206:

$$T(z,t) = \Delta T \exp\left(-z\sqrt{\frac{\omega}{2a}}\right)\cos\left(\omega t - z\sqrt{\frac{\omega}{2a}}\right) - \frac{2}{\sqrt{\pi}}\int_0^{z/\sqrt{4at}} \mathrm{e}^{-\eta^2}\cos\left[\omega\left(t - \frac{x^2}{4a\eta^2}\right)\right]\mathrm{d}\eta \tag{19}$$

where η is independent variable defined as:

$$\eta = \frac{z}{\sqrt{4a(t-\tau)}} \tag{20}$$

Solution in eq. (19) contains transition part, the last term in equation, hence it covers any time point starting from $t = 0$. This feature adds to the generality of solution and distinguishes it from other quasi steady-state solutions commonly found in literature. However, this transition effect diminishes as t increases and process reaches steady-state regime.

Ground surface heat flux, for surface temperature time dependant function as defined in eq. (18), can be derived after putting eq. (19) in eq. (3). Upon the differentiation with respect to z, ground surface heat flux becomes:

$$q(0,t) = \Delta T \lambda \sqrt{\frac{\omega}{a}}\cos\left(\omega t + \frac{\pi}{4}\right) \tag{21}$$

Solution does not contain derivative of the last term in eq. (19), hence it covers only the cases where time is sufficiently large, that is when transition effect diminishes.

Case 2 – Sinusoidal ground surface temperature model

In contemporary soil science literature, sinusoidal functions are most frequently used for ground surface temperature fluctuation approximation [1, 2, 21, 22, 25]. As proposed by Hiller [21], the surface temperature can have the following form:

$$\mathrm{f}(\tau) = T(0,t) = T_0 + \Delta T \sin(\omega t), \quad t > 0 \tag{22}$$

where $T_0 = T(z, 0)$ and ΔT as well as ω have the same values as defined in eq.(18). Steady-state thermal regime for surface temperature defined as in eq. (22) gets the following form [26]:

$$T(z,t) = T_0 + \Delta T \exp\left(-z\sqrt{\frac{\omega}{2a}}\right)\sin\left(\omega t - z\sqrt{\frac{\omega}{2a}}\right) \tag{23}$$

Making use of eq. (3), ground heat flux becomes:

$$q(z,t) = -\lambda\frac{\partial T(z,t)}{\partial z} = \Delta T\lambda\sqrt{\frac{\omega}{2a}}\exp\left(-z\sqrt{\frac{\omega}{2a}}\right)\left\{\sin\left(\omega t - z\sqrt{\frac{\omega}{2a}}\right) + \cos\left(\omega t - z\sqrt{\frac{\omega}{2a}}\right)\right\} \tag{24}$$

or in simplified form:

$$q(z,t) = -\Delta T\lambda\sqrt{\frac{\omega}{2a}}\exp\left(-z\sqrt{\frac{\omega}{2a}}\right)\sin\left(\omega t - z\sqrt{\frac{\omega}{2a}} + \frac{\pi}{4}\right) \tag{25}$$

Clearly, setting $z = 0$ gives the ground surface heat flux:

$$q(0,t) = \Delta T\lambda\sqrt{\frac{\omega}{a}}\sin\left(\omega t + \frac{\pi}{4}\right) \tag{26}$$

The solution in eq. (26) is the same as the result obtained in [27].

Diffusion equation for heat propagation in ground – fractional approach

The subject of fractional calculus is rather old, dating back to Leibnitz and almost coinciding with time of his work on classical calculus. In his letter to L'Hopital, dated September 30, 1695, Leibnitz for the first time provided note and meaning of the derivative of order of one half [28]. His idea was further elaborated and got more or less finished form primarily due to Liouville, Letnikov, and Riemann. From that time on, fractional derivatives have been developed solely as pure theoretical field without any notable contribution to practice.

One of the reasons for this was the lack of geometric interpretation of fractional derivative, which was not the case with integer derivatives. However, inability of classical, integer order derivative models in explaining complex phenomena (especially in elastodynamics, material science, electrochemistry, chemical physics, and rheology), propelled further research in field and demonstrated strength of fractional calculus in solving practical problems. Some of the early works can be found in [29-33].

Power of fractional calculus in solving diffusion related problems was originally demonstrated by Oldham *et al*. [34]. Later on, application of fractional calculus was extended to heat transfer (more particularly to conduction) problems. Kulish [16] demonstrated straightforwardness of fractional calculus procedure for solving conduction related problems in semi-infinite and 1-D case where the task was in determining surface time-varying temperature for given transient heat flux and vice versa. In fractional calculus approach, idea is in reducing the order of initial differential equation and consequently the computation time.

Assuming that the heat propagation through ground is 1-D, and that the vertical dimension starting from the ground surface to the earth centre can be approximated as infinite (reasonable assumption taking into account: significant dumping effect of soil on temperature wave propagation and ratio of ground depth of interest and magnitude of earth diameter), fractional calculus procedure should be used with confidence to this kind of problems. This notion was firstly proposed by Wang *et al*.[15].

Here, procedure for developing fractional-diffusion equation for heat propagation in ground will be based on the work of Agrawal [17] and Kulish [16]. Let us start with the eq. (6) and initial eq. (7) as well as boundary conditions (8) and (9). Introducing the auxiliary variable Φ, defined as:

$$\Phi = T - T_0 \tag{27}$$

and applying the Laplace transformation of eq. (6) and boundary condition (8) gives:

$$\frac{\partial^2 \Phi(z,s)}{\partial z^2} - \frac{s}{a}\Phi(z,s) = 0 \tag{28}$$

$$\Phi(z, s) = F(s) \text{ for } z = 0 \tag{29}$$

The solution of eq. (28) can be expressed as:

$$\Phi(z,s) = A\exp\left(z\sqrt{\frac{s}{a}}\right) + B\exp\left(-z\sqrt{\frac{s}{a}}\right) \tag{30}$$

Taking into account boundary condition (9), which can be expressed in s-domain as, $\Phi(\infty, s) = 0$, constant A in eq. (30) turns into zero. Hence, eq. (30) becomes:

$$\Phi(z,s) = B\exp\left(-z\sqrt{\frac{s}{a}}\right) \tag{31}$$

Provided that $\Phi(z, s) = F(s)$ for $z = 0$, according to boundary conditions, one has:

$$B = \Phi(s) \tag{32}$$

Substituting eq. (32) in eq. (31):

$$\Phi(z,s) = \Phi(s)\exp\left(-z\sqrt{\frac{s}{a}}\right) \tag{33}$$

and differentiating eq. (33) with respect to z gives:

$$\frac{\partial \Phi(z,s)}{\partial z} = -\sqrt{\frac{s}{a}}\Phi(s)\exp\left(-z\sqrt{\frac{s}{a}}\right) = -\sqrt{\frac{s}{a}}\Phi(z,s) \tag{34}$$

Making use of inverse Laplace theorem eq. (34) becomes:

$$\frac{\partial \Phi(z,t)}{\partial z} = \frac{1}{L}\left(-\sqrt{\frac{s}{a}}\Phi(z,s)\right) \tag{35}$$

Here, following property [35]:

$$L[D^{1/2}\mathrm{f}(t)] = \sqrt{s}F(s) \tag{36}$$

can be used for switching to half-order time derivative, where $D^{1/2}$f(t) can be defined according to Riemann-Liouville notion as [35]:

$$\mathrm{D}^{\alpha}\mathrm{f}(t) = \frac{\partial^{\alpha}[\mathrm{f}(t)]}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\mathrm{df}(t)}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{(t-\tau)^{\alpha}} \tag{37}$$

Applying property from eq.(36) and Laplace transform rules, eq. (37) becomes:

$$-\frac{\partial \Phi(z,t)}{\partial z}=\frac{1}{\sqrt{a}}\frac{\partial^{1/2}\Phi(z,t)}{\partial t^{1/2}} \tag{38}$$

or after restoring the original variables:

$$-\frac{\partial T(z,t)}{\partial z}=\frac{1}{\sqrt{a}}\left[\frac{\partial^{1/2}T(z,t)}{\partial t^{1/2}}-\frac{\partial^{1/2}T(z,0)}{\partial t^{1/2}}\right] \tag{39}$$

Evoking (3) and using the properties of differintegration [34], heat flux becomes:

$$q(z,t)=\frac{\lambda}{\sqrt{a}}\left[\frac{\partial^{1/2}T(z,t)}{\partial t^{1/2}}-\frac{T_0}{\sqrt{\pi t}}\right] \tag{40}$$

Applying definition of fractional derivative from eq. (37), eq. (40) leads to:

$$q(z,t)=\frac{\lambda}{\sqrt{a}}\left[\frac{1}{\Gamma(1/2)}\int_0^t\frac{\mathrm{d}T(z,\tau)}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\sqrt{t-\tau}}-\frac{T_0}{\sqrt{\pi t}}\right] \tag{41}$$

Ground heat flux on the surface can be simplified from eq. (40), setting $z = 0$:

$$q(0,t)=\frac{\lambda}{\sqrt{a}}\left[\frac{\partial^{1/2}T(0,t)}{\partial t^{1/2}}-\frac{T_0}{\sqrt{\pi t}}\right] \tag{42}$$

In order to verify the procedure, fractional solutions for ground surface heat flux provided in eq. (42) were computed for ground surface temperatures as defined in eqs. (18) and (27) and compared with analytic solutions.

Case 1 – Cosine ground surface temperature model – fractional approach

Provided that the ground surface temperature is defined as in eq. (18), eq. (42) obtains the following form:

$$q(0,t)=\frac{\lambda}{\sqrt{a}}\left[\frac{\partial^{1/2}[\Delta T\cos(\omega t)}{\partial t^{\,1/2}}-\frac{T_0}{\sqrt{\pi t}}\right] \tag{43}$$

Making use of semiderivative of cos(t) [34]:

$$\frac{\partial^{1/2}[\cos(t)]}{\partial t^{1/2}}=\frac{1}{\sqrt{\pi t}}+\cos\left(t+\frac{\pi}{4}\right)-\sqrt{2}A\sqrt{\frac{2t}{\pi}} \tag{44}$$

and following properties [34]:

$$\frac{\partial^{1/2}[C\mathrm{f}(x)]}{\partial t^{1/2}}=C\frac{\partial^{1/2}[\mathrm{f}(x)]}{\partial t^{1/2}} \tag{45}$$

$$\frac{\partial^{1/2}[\mathrm{f}(Ct)]}{\partial t^{1/2}}=\sqrt{C}\frac{\partial^{1/2}[\mathrm{f}(Ct)]}{\partial (Ct)^{1/2}} \tag{46}$$

eq. (43) becomes:

$$q(0,t)=\lambda\Delta T\sqrt{\frac{\omega}{a}}\left[\frac{1}{\pi\sqrt{\omega t}}+\cos\left(\omega t+\frac{\pi}{4}\right)-\sqrt{2}A\sqrt{\frac{2\omega t}{\pi}}-\sqrt{\frac{a}{\omega}}\frac{T_0}{\sqrt{\pi t}}\right] \tag{47}$$

where A is auxiliary Fresnel integral defined as:

$$A(x)=\left[\frac{1}{2}-S(x)\right]\cos\left(\frac{\pi x^2}{2}\right)-\left[\frac{1}{2}-C(x)\right]\sin\left(\frac{\pi x^2}{2}\right) \tag{48}$$

with functions $S(x)$ and $C(x)$ defined as:

$$C(x)=\int_0^x\cos\left(\frac{\pi t^2}{2}\right)\mathrm{d}t \quad \text{and} \quad S(x)=\int_0^x\sin\left(\frac{\pi t^2}{2}\right)\mathrm{d}t \tag{49}$$

For large enough time, as process reaches the steady-state regime, value of Fresnel function (48) becomes negligible hence the third term in eq. (47) can be omitted. Similarly, the first term becomes insignificant. Finally, assuming that heat flux is close to zero in initial moment, eq. (47) obtains exactly the same form as in eq. (21).

Case 2 – Sinusoidal ground surface temperature model – fractional approach

As to the sinusoidal ground surface temperature model, approach is similar. Provided that the ground surface temperature variation model is defined as in eq. (22), ground surface heat flux, upon substitution of eq. (22) in eq. (47), obtains the following form:

$$q(0,t)=\frac{\lambda}{\sqrt{a}}\left[\frac{\partial^{1/2}[T_0+\Delta T\sin(\omega t)]}{\partial t^{1/2}}-\frac{T_0}{\sqrt{\pi t}}\right] \tag{50}$$

or in simplified form:

$$q(0,t)=\frac{\lambda}{\sqrt{a}}\frac{\partial^{1/2}[\Delta T\sin(\omega t)]}{\partial t^{1/2}} \tag{51}$$

Making use of semiderivative of $\sin(x)$ [34]:

$$\frac{\partial^{1/2}[\sin(t)]}{\partial t^{1/2}}=\sin\left(t+\frac{\pi}{4}\right)-\sqrt{2}B\sqrt{\frac{2t}{\pi}} \tag{52}$$

and properties of semiderivative stated in eqs. (45) and (46), ground surface heat flux gets the final form:

$$q(0,t)=\lambda\Delta T\sqrt{\frac{\omega}{a}}\left[\sin\left(\omega t+\frac{\pi}{4}\right)-\sqrt{2}B\sqrt{\frac{2\omega t}{\pi}}\right] \tag{53}$$

where B is auxiliary Fresnel integral defined as:

$$B(x)=\left[\frac{1}{2}-C(x)\right]\cos\left(\frac{\pi x^2}{2}\right)+\left[\frac{1}{2}-S(x)\right]\sin\left(\frac{\pi x^2}{2}\right) \tag{54}$$

and $C(x)$ and $S(x)$ are defined as in eq. (49).

Similarly as in the case with cosine ground surface temperature function, for large enough time, value of Fresnel function (54) becomes negligible hence the last term in eq. (53) can be omitted. Upon performing this simplification, ground surface heat flux gets exactly the same form as in eq. (26).

Conclusions

It has been shown that solutions of problems of heat conduction in ground could be considerably simplified with application of fractional (half order time) derivatives. Results indicate that solutions obtained with fractional calculus closely correspond to analytic solutions with remark that fractional calculus solutions are more general containing the term covering the transitional effect. Method is particularly effective in determining surface heat flux for a given surface temperature time dependant function. In that case solution reduces to single eq. (48) and consequently to half-order time derivative of given ground surface temperature function. While procedures for determining analytical solutions for arbitrary surface temperature functions exist, primary owing to Duhamel theorem, they are intricate and very often require considerable computation effort. This particularly stands for complex ground surface time dependant temperature functions.

Cases studied in this article cover only the most frequently encountered models of ground surface temperature oscillations; however approach can be extended with confidence to one containing more complex functions. This is of particular importance taking into account recently published papers [22, 36] suggesting and demonstrating, respectively, inability of simple harmonic functions in providing accurate estimates of ground temperature variations. This kind of problems could be overcome using the data-driven models – developed from representative time series obtained from measurement (*e. g*. remote sensing), where artificial neural networks with appropriate architecture seems promising in providing satisfactory estimation results. Consequently, obtained surface temperature models could be used in estimating unknown ground surface heat flux, using developed fractional derivative equations.

Finally, it can be concluded that obtained results encourage further work and in the same time indicate that application of fractional calculus should not be confined merely to soil sciences. Furthermore, method could be easily extended to provide solutions to similar problems encountered in hydrology, environmental and energy related disciplines.

Nomenclature

a – coefficient of thermal diffusivity, [m^{2}s^{-1}]
c – soil specific heat capacity per unit mass, [Jkg^{-1}K^{-1}]
erfc – error complementary function
f(τ) – arbitrary surface temperature – time-dependent function
q – ground heat flux, [Wm^{-2}]
t, τ – time, [s]
T_0 – initial temperature of the surface, [K]
DT – amplitude of the surface temperature fluctuation, [K]
∇T – spatial gradient of temperature, [K]

Greek symbols

l – thermal conductivity, [W m^{-1}K^{-1}]
ρ – soil density, [kgm^{-3}]
Φ – auxiliary variable

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