ELASTIC-PLASTIC TRANSITION STRESSES IN ROTATING CYLINDER BY FINITE DEFORMATION UNDER STEADY-STATE TEMPERATURE

by

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The problem of elastic-plastic transition stresses of thick walled rotating cylinder by finite deformation under steady-state temperature has been solved by using the concept of generalised strain measure. It has been seen that with the increase of temperature, the cylinder having smaller radii ratios requires lesser angular velocity to become fully plastic as compared to cylinder having higher radii ratios. The result for the combined effect of rotation and temperature are calculated for fully plastic and depicted graphically.

Key words: *elastic, plastic, stresses, cylinder, temperature*

Introduction

Rotating cylinder play an important role in the machine designs. The problem of a uniformly long thick-walled circular cylinder arises occasionally in the design of turbine rotors. The problem of thick-walled rotating cylinders without thermal effect has been discussed by Divis *et al*. [1], and Rimrott [2] in plastic theory. In analysing the problem, these authors assumed incompressibility of the material, yield condition, and relationship between stress and strain. Incompressibility of the material in plasticity problems is an assumption which simplifies the problem. In fact it is not possible to find a solution in closed from without this assumption. Seth's transition theory does not require any of these assumptions like yield condition, incompressibility condition and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. This theory utilises the concept of generalized strain measure which not only gives the well known strain measure but can also be used to find the stresses in plasticity and creep problems by determining the asymptotic solution at transition points of the governing equations. Seth [3] has defined the generalized principal strain measure as:

$$e_{ii} = \int_0^{e_{ii}^A} \left[1 - 2e_{ii}^A\right]^{(n/2)-1} \mathrm{d}e_{ii}^A = \frac{1}{n}\left[1 - (1 - 2e_{ii}^A)^{n/2}\right], \quad (i = 1, 2, 3) \tag{1}$$

where n is the measure.

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In this paper, we investigate the problem of elastic-plastic transition stresses in rotating cylinder by finite deformation under steady-state temperature, by using Seth's transition theory. Results have been discussed numerically and depicted graphically.

Governing equations

Let us consider a thick-walled circular cylinder of internal radius a and external radius b, rotating about its axis with an angular velocity ω and subjected to a steady-state temperature Θ at the inner surface $r = a$. The displacement components in cylindrical polar co-ordinate are given by [4]:

$$u = r(1-\beta); \quad v = 0; \quad w = dz \tag{2}$$

where β is function of $r = (x^2 + y^2)^{1/2}$ only and d is a constant.

The strain components for finite deformation are given by [3]:

$$\begin{aligned} e_{ii}^A &= \frac{1}{2}\left[1- (r\beta'+\beta)^2\right] \\ e_{\theta\theta}^A &= \frac{1}{2}\left[1-\beta^2\right] \\ e_{zz}^A &= \frac{1}{2}\left[1-(1-d)^2\right] \\ e_{r\theta}^A &= e_{\theta z}^A = e_{zr}^A = 0 \end{aligned} \tag{3}$$

where $\beta' = \mathrm{d}\beta/\mathrm{d}r$ and meaning of superscripts A is Almansi

By substituting eq. (3) in eq. (1), the generalized components of strain become:

$$\begin{aligned} e_{rr} &= \frac{1}{n}\left[1-(r\beta'+\beta)^n\right] \\ e_{\theta\theta} &= \frac{1}{n}[1-\beta^n] \\ e_{zz} &= \frac{1}{n}[1-(1-d)^n] \\ e_{r\theta} &= e_{\theta z} = e_{zr} = 0 \end{aligned} \tag{4}$$

The stress-strain relation for thermo-elastic isotropic material are given by [5]:

$$T_{ij} = \lambda\delta_{ij}I_1 + 2\mu e_{ij} - \xi\Theta\delta_{ij}, \quad (i, j = 1, 2, 3) \tag{5}$$

where T_{ij} are the stress components, λ and μ are Lame's constants, $I_1 = e_{kk}$ is the first strain invariant, δ_{ij} – the Kronecker delta, $\xi = \alpha(2\lambda + 2\mu)$, α – the coefficient of thermal expansion, and Θ – the temperature. Further, Θ has to satisfy:

$$\nabla^2\Theta = 0$$

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}^2\Theta}{\mathrm{d}r^2} \equiv \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}\Theta}{\mathrm{d}r} = 0$$

or

$$\frac{\mathrm{d}\Theta}{\mathrm{d}r} = \frac{k}{r}$$

which has the following solution:

$$\Theta = k_1 \log r + k_2 \tag{6}$$

where k_1 and k_2 are constant of integration, which can be determined from the boundary condition.

By substituting eq. (4) in eq. (5), the stresses are obtained as:

$$\begin{aligned} T_{rr} &= \frac{\lambda + 2\mu}{n}\left[1 - (r\beta' + \beta)^n\right] + \lambda k + \frac{\lambda}{n}(1 - \beta^n) - \xi\Theta \\ T_{\theta\theta} &= \frac{\lambda}{n}\left[1 - (r\beta' + \beta)^n\right] + \lambda k + \frac{\lambda + 2\mu}{n}(1 - \beta^n) - \xi\Theta \\ T_{zz} &= \frac{\lambda}{n}\left[1 - (r\beta' + \beta)^n\right] + (\lambda + 2\mu)k + \frac{\lambda}{n}(1 - \beta^n) - \xi\Theta \\ T_{r\theta} &= T_{\theta z} = T_{zr} = 0 \end{aligned} \tag{7}$$

where $\beta' = \mathrm{d}\beta/\mathrm{d}r$ and k = $[1 - (1 - d)^n]/n$.

The equations of equilibrium are all satisfied except:

$$\frac{\mathrm{d}T_{rr}}{\mathrm{d}r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \rho\omega^2 r^2 = 0 \tag{8}$$

where ρ is the density of the material.

The temperature field satisfying eq. (6) and

$$\theta = \theta_0 \text{ at } r = a, \quad \theta = 0 \text{ at } r = b.$$

where θ_0 is constant, given by:

$$\theta = \frac{\theta_0 \log \frac{r}{b}}{\log \frac{a}{b}} \tag{9}$$

By substituting eqs. (9) and (12) in eq. (10), one gets the non-linear differential equation:

$$\left[\left(P + \frac{c}{n}\right)(P+1)^n + (1-c)P - \frac{c}{n} + \frac{c\xi\overline{\Theta_0}}{2\mu\beta^n} - \frac{\rho\omega^2 r^2 c}{2\mu\beta^n}\right] + \beta P (P+1)^{n-1}\frac{\mathrm{d}P}{\mathrm{d}\beta} = 0 \tag{10}$$

where c is the compressibility factor of the material in term of Lame's constant, given by: $c = 2\mu/\lambda + 2\mu$ and $\overline{\theta_0} = \theta_0/\log(a/b)$.

Transition points of β in eq. (10) are $P \to \pm\infty, -1$.

The boundary conditions require that:

(1) $$T_{rr} = 0 \quad \text{at} \quad r = a \quad \text{and} \quad r = b \tag{11}$$

(2) The resultant force normal to the plane $z =$ constant must vanish, *i. e.*,

$$\int_a^b rT_{zz}\mathrm{d}r = 0 \tag{12}$$

Solution through the principal stress

It has been shown [6-14] that the asymptotic solution through the principal stress lead from elastic state to plastic state at the transition point $P \to \pm\infty$. If the transition function R is defined as:

$$R = 1 - \frac{nT_{rr}}{3\lambda + 2\mu} - \frac{nc\xi\Theta}{3\lambda + 2\mu} - \frac{n\rho\omega^2 r^2}{6\lambda + 4\mu} \equiv$$

$$\equiv \frac{1}{3-2c}\left[(1-c)\beta^n + (P+1)^n\beta^n + (1-c)(1-d)^n + \frac{n(1-c)\xi\Theta}{\lambda + 2\mu} - \frac{n\rho\omega^2 r^2}{2(\lambda + 2\mu)}\right] \tag{13}$$

and by taking the logarithmic differentiation of eq. (13) with respect to r and by using eq. (10), one gets:

$$\frac{\mathrm{d}(\log R)}{\mathrm{d}r} = \frac{n\beta^n\left\{\frac{c}{n}[1-(P+1)^n] - \frac{-c^2\xi\bar{\Theta}_0}{2\mu\beta^n}\right\} + n\beta P(2-c)\sqrt{(2\beta-1)^n}}{r\left\{[(1-c)+(P+1)^n]\beta^n + +\frac{n(1-c)\xi\Theta}{\lambda + 2\mu} - \frac{n\rho\omega^2 r^2}{2(\lambda + 2\mu)}\right\}} \tag{14}$$

By taking the asymptotic value of eq. (14) as $P \to \pm\infty$, one gets:

$$\frac{\mathrm{d}(\log R)}{\mathrm{d}r} = -\frac{c}{r} \tag{15}$$

By integrating eq. (15), one gets:

$$R = L_0 r^{-c} \tag{16}$$

where L_0 is the constant of integration.

From eqs. (16) and (13), one gets:

$$T_{rr} = \frac{3\lambda + 2\mu}{2n}(1 - L_0 r^{-c}) - \frac{\rho\omega^2 r^2}{2} - c\xi\Theta \tag{17}$$

The value of E in the transition range is given by [4]:

$$Y = \frac{E}{n} = \frac{2\mu(3-2c)}{n(2-c)} \tag{18}$$

where Y is the yield stress in tension.

By substituting eq. (18) in eq. (17), one gets:

$$T_{rr} = (2-c)\frac{Y}{c}(1 - L_0 r^{-c}) - \frac{\rho\omega^2 r^2}{2} - c\xi\Theta \tag{19}$$

where $1 - L_0 r^{-c} \to 0$ as $L_0 \to 1$.

By substituting of eq. (19) in eq. (8), one gets:

$$T_{\theta\theta} = T_{rr} + (2-c)YL_0 r^{-c} - \frac{\rho\omega^2 r^2}{2} \tag{20}$$

Now, eqs. (7) give:

$$T_{zz} = \frac{1-c}{2-c}(T_{rr} + T_{\theta\theta}) + 2K_1 - \frac{c\xi\Theta}{2-c} \tag{21}$$

where $K_1 = \dfrac{\mu(3-2c)e_{zz}}{2-c}$.

By applying boundary conditions (11) and (12) in eqs. (19) and (21), one gets:

$$L_0 = a^c\left[1 - \frac{c^2\xi\Theta_0}{(2-c)Y} - \frac{\rho\omega^2 a^2 c}{(4-2c)Y}\right] \tag{22}$$

and

$$\frac{\rho\omega^2}{2} = \frac{c\xi\Theta_0 b^{-c} - \dfrac{(2-c)Y(b^{-c}-a^{-c})}{c}}{a^{-c}b^2 - a^2 b^{-c}} \tag{23}$$

$$K_1 = \frac{c^2\xi\overline{\Theta_0}\left(a^2 - b^2 - 2a^2\log\dfrac{a}{b}\right)}{4(2-c)(b^2-a^2)} - \frac{(1-c)\rho\omega^2(a^2+b^2)}{8-4c} \tag{24}$$

By substituting L_0 and K_1 into eqs. (19)-(21), one gets the transitional stresses.

Since $|T_{\theta\theta} - T_{rr}|$ is maximum at $r = a$, the yielding of the cylinder will take place at the internal surface. In this case eq. (20) gives:

$$|T_{\theta\theta} - T_{rr}| = \left[(2-c)Y - c^2\xi\Theta_0 - 0.5c\rho^2 a^2\omega^2\right] - c\xi\bar{\Theta}_0 \equiv Y_1 \tag{25}$$

By substituting the value of Y in terms of Y_1 into eqn. (23), one gets a relation between angular velocity ω and temperature Θ_0 as:

$$\frac{\rho\omega^2}{2} = \frac{c\xi\bar{\Theta}_0 b^{-c} - (b^{-c} - a^{-c})\left\{Y_1 + c^2\xi\bar{\Theta}_0 + \dfrac{0.5\rho\omega^2 a^2 c + c\xi\bar{\Theta}_0}{c}\right\}}{a^{-c}b^2 - a^2 b^{-c}} \tag{26}$$

The stresses for fully plastic state are obtained by taking $c \to 0$ in eqs. (25), (26), (19), (20), and (21) as:

$$|T_{\theta\theta} - T_{rr}| = 2Y - 2\alpha E\Theta_0 \equiv Y_2 \tag{27}$$

$$T_{rr} = Y_2\log\frac{r}{a} + \frac{\rho\omega^2(a^2 - r^2)}{2} \tag{28}$$

$$T_{\theta\theta} = Y_2\left[\log\frac{r}{a} + 1\right] + \frac{\rho\omega^2(a^2 - r^2)}{2} \tag{29}$$

$$T_{zz} = \frac{1}{2}\; T_{rr} + T_{\theta\theta} \;\; + \frac{2K_1}{b^2 - a^2} - \frac{\rho\omega^2(b^2 + a^2)}{8} - \alpha E\Theta \tag{30}$$

where

$$K_1 = \frac{\alpha E\Theta_0 \left(a^2 - b^2 - 2a^2 \log\dfrac{a}{b} \right)}{4} \tag{31}$$

$$\frac{\rho\omega^2}{2} = Y_2 \frac{\log\dfrac{b}{a}}{b^2 - a^2} \tag{32}$$

When there is no thermal effect *i. e.* $\Theta_0 = 0$, eqs. (30)-(32) are the same as obtained by Gupta [13] for hollow rotating cylinders.

Numerical illustration and discussion

In fig. 1, curves have been drawn between $\omega*/Y^*$ and b_0/Y^* to present yielding through the whole of the cylinder (fully plastic state) for different wall thickness ratios. It has been seen that with the increase of temperature, the cylinder having smaller radii ratios requires lesser angular velocity to become fully plastic as compared to cylinder having higher radii ratios. The strain in the longitudinal directions as a function of temperature and the corresponding rotational speed (from fig. 1) is plotted in fig. 2, for cylinder having different thickness ratios. In the absence of thermal effects, the axial contraction increases with the increase in thickness ratios of the cylinder, but with the inclusion of temperature effects, it is seen that cylinder having thickness ratios 2 has greater axial contraction than cylinders having thickness ratios 3, 4, 7, and 10. It is of interest to note that at higher temperature axial contraction become greater for *b/*2 = 2, then decreases for *b/a* = 3, remains almost the same for *b/a* = 4, then again increases for cylinders having thickness ratio *b/a* > 4.

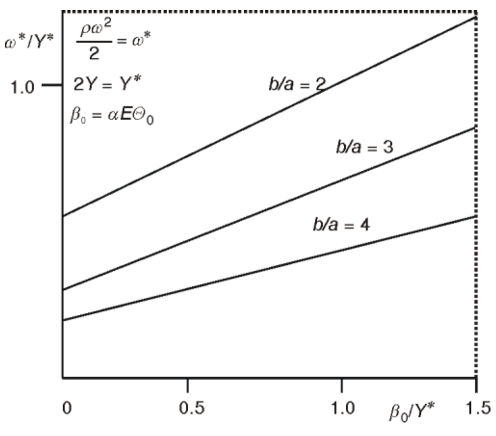

Figure 1. Relation between $\omega*/Y^*$ and b_0/Y^* for yielding through the whole cylinder for various thickness ratios

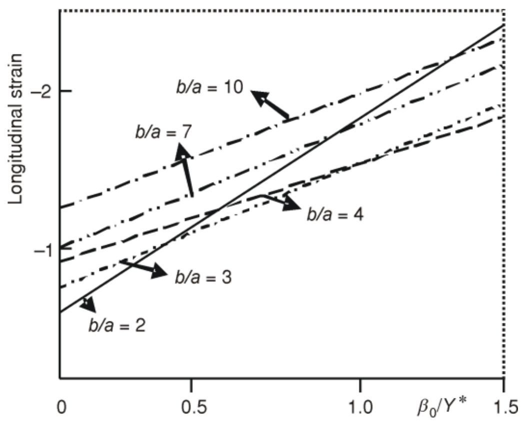

Figure 2. Plots of longitudinal strain *vs.* temperature for various cylinder ratios

Conclusion

It has been seen that with the increase of temperature, the cylinder having smaller radii ratios requires lesser angular velocity to become fully plastic as compared to cylinder having higher radii ratios.

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Nomenclature

A – Alamansi finite strain components, [–]
a, b – internal and external radii of the circular cylinder, [m]
c – compressibility factor, [–]
E – modulus of elasticity, [Pa]
n – measure, [m]
P, K_1 – constant, [–]
T_{ij}, e_{ij} – stress strain rate tensors, [kgm^{-1}s^{-2}]
T_{rr}/E – radial stress component, [–]
$T_{\theta\theta}/E$ – circumferential stress component, [–]
u, v, w – displacement components, [m]
Y – yield stress, [kgm^{-1}s^{-2}]

Greek symbols

β – function of *r* only, [m]
λ, μ – Lame's constants, [–]
ρ – density of material, [kgm^{-3}]
Θ – temperature, [K]

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