A NOTE ON THE HOMOTOPY PERTURBATION METHOD

by

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The homotopy perturbation method admits some unknown parameters in the obtained series solutions, which can be identified after few iteration steps using the method of least squares. The solution procedure of the so-called optimal homotopy asymptotic method follows the same way.

Key words: homotopy equation, parameter-expansion method, method of least squares

Babaelahi, Ganji, and Joneidi applied the so called Optimal Homotopy Asymptotic Method [1] to solve the problem of steady incompressible mixed convection flow past vertical flat plate [2]. The method is very effective to solve non-linear equations, but this method can not be considered as a new method, it is actually a coupled method of the homotopy perturbation method [3-5] and the method of least squares. The basic idea was systematically presented in 2007 International Symposium of Nonlinear Dynamics, October 27-30, 2007, Shanghai, China, and was then published in Topological Methods in Non-Linear Analysis [6]. I also gave this basic idea in ScienceWatch, see http://sciencewatch.com/inter/aut/2008- apr/08aprHe, especially the fifth solution procedure “Optimal identification of the unknown parameter in the trial function”.

Hereby I will illustrate the general solution procedure of the homotopy perturbation method [3-5]. Consider a non-linear equation in the form:

\[ Lu + Nu = 0 \]  (1)

where \( L \) and \( N \) are linear operator and non-linear operator, respectively. In order to use the homotopy perturbation method, a suitable construction of a homotopy equation is of vital importance. Generally a homotopy can be constructed in the form:

\[ \tilde{L}u + p(Lu + Nu - \tilde{L}u) = 0 \]  (2)

where \( \tilde{L} \) can be a linear operator or a simple non-linear operator, and the solution of \( \tilde{L}u = 0 \) with possible some unknown parameters can basically describe the original non-linear system. For example for a non-linear oscillator we can choose \( \tilde{L}u = u'' + \omega^2 u \), where \( \omega \) is the frequency of the non-linear oscillator. We use a simple example to illustrate the solution procedure.

As an example, we consider a nonlinear oscillator:

\[ u'' + u^3 = 0, \quad u(0) = A, \quad u'(0) = 0 \]  (3)

We can construct the following homotopy eq. (4):

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where \( \omega \) is the frequency of the oscillator which is to be solved.

When \( p = 0 \), we have \( u'' + \omega^2 u = 0 \), and we have the solution \( u = A \cos \omega t \). If \( \omega \) can be optimally identified, then even the zeroth order solution \( u = A \cos \omega t \) can excellently describe the oscillation property of eq. (3). When \( p = 1 \), eq. (4) turns out to be the original one, eq. (3). We assume that the solution can be expressed in the form:

\[
u = u_0 + pu_1 + p^2u_2 + \ldots
\]

We write down the linear equations for \( u_0 \) and \( u_1 \):

\[
u_0'' + \omega^2 u_0 = 0
\]

\[
u_1'' + \omega^2 u_1 + u_0^3 - \omega^2 u_0 = 0
\]

The standard homotopy perturbation method always stops before the second iteration.

For a beginner, the homotopy equation can be constructed in a simple way, that is:

\[
Lu + pNu = 0
\]

For eq. (3), the homotopy equation reads:

\[
u'' + pu^3 = 0
\]

This will lead to an infinite series solution which converges to the exact solution. In order to minimize the solution procedure, we re-write eq. (9) in the form [7]:

\[
u'' + 0u + pu^3 = 0
\]

Using the parameter-expansion method [8-14], the coefficient, 0, of the linear term is also expanded into a series in \( p \), that is:

\[
0 = \omega^2 + p\alpha_1 + p^2\alpha_2 + \ldots
\]

where \( \omega \) and \( \alpha_i \) are unknown constants to be further determined.

Substituting eqs.(5) and (11) into eq.(10), we have:

\[
u_0'' + \omega^2 u_0 = 0
\]

\[
u_1'' + \omega^2 u_1 + a_1u_0 + u_0^3 - \omega^2 u_0 = 0
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\]

If the first-order approximate solution is solved, then from eq. (11), by setting \( p = 1 \), we have:

\[
\omega^2 + a_1 = 0
\]

It is obvious that eqs. (13) and (14) are exact same with eqs. (6) and (7).

If a higher order approximate solution is solved, the unknown can be solved using the method of least squares [6]:

\[
\int_0^b R \frac{\partial R}{\partial a_i} \, dt = 0
\]
where \( T = \frac{2\pi}{\omega}, R \) is the residual of eq. (1) when the iteration procedure stop, \( R = L u_0 + N u_x \).

Now we turn back to the solution procedure of the optimal homotopy asymptotic method [1, 2], where a homotopy equation is constructed in the form:

\[
Lu + H(p)Nu = 0
\]  

(17)

where \( H(p) \) is expanded in the form

\[
H(p) = pc_1 + p^2c_2 + ...
\]  

(18)

and the unknown constants \( c_1 \) is identified similar to eq. (16), see ref. [1, 2]. This solution procedure is effective and only few iteration steps are needed.

As I said before, this can not be considered as a new method, the main difference lies in construction of the homotopy equation.

We can write the homotopy equation, eq. (8), in the form:

\[
Lu + 1 \cdot pNu = 0
\]  

(19)

Expanding the constant,1, into a series in \( p \):

\[
1 = pc_1 + p^2c_2 + ...
\]  

(20)

The first terms of the obtained series are exactly same with those by the optimal homotopy asymptotic method.

In my previous publication [3], I suggested that a homotopy equation can be also constructed in the form:

\[
(1 - p)(Lv - Lu_0) + pNv = 0
\]  

(21)

where \( u_0 \) is the initial solution which must outline the basic solution property and it can involve some unknown parameters which can be identified after few iterations by the method of least squares.

For the problem in [2]:

\[
f''' + \frac{f''}{2} + R_i T = 0, \quad f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1
\]  

(22)

\[
T'' + \frac{Pr}{2} fT' = 0, \quad T(0) = 1, \quad T(\infty) = 0
\]  

(23)

we can construct the following homotopy equations:

\[
(1 - p)(u''' - f_0''' - f_0') + p(u'' + \frac{uu''}{2} + R_i v) = 0
\]  

(24)

\[
(1 - p)(v'' - T_0'') + p(v' + \frac{Pr}{2} uv') = 0
\]  

(25)

There are alternative approaches to construction of homotopy equations for eqs. (22) and (33), the beginners are recommended to read [15, 16].

The initial solutions can be chosen as follows:

\[
f_0 = \frac{1}{a}(e^{-a\eta} - 1) + \eta
\]  

(26)

\[
T_0 = e^{-\eta}(1 + b\eta)
\]  

(27)

where \( a \) and \( b \) are unknown constants to be further determined.
It is obvious that eqs. (26) and (27) satisfy the boundary/initial conditions, the unknown parameters $a$ and $b$ in eqs. (26) and (27) can be identified using the method of least squares after few iterations. The solution procedure is of course much simpler than that in [2].

Generally the method of least squares can be used for optimal identification of unknown parameters involved in a series solution. Recently Herisanu et al. suggested an optimal variational iteration algorithm based on the variational iteration method and the method of the least squares [17].

References