HEAT-BALANCE INTEGRAL TO FRACTIONAL (HALF-TIME) HEAT DIFFUSION SUB-MODEL

by

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The fractional (half-time) sub-model of the heat diffusion equation, known as Dirac-like evolution diffusion equation has been solved by the heat-balance integral method and a parabolic profile with unspecified exponent. The fractional heat-balance integral method has been tested with two classic examples: fixed temperature and fixed flux at the boundary. The heat-balance technique allows easily the convolution integral of the fractional half-time derivative to be solved as a convolution of the time-independent approximating function. The fractional sub-model provides an artificial boundary condition at the boundary that closes the set of the equations required to express all parameters of the approximating profile as function of the thermal layer depth. This allows the exponent of the parabolic profile to be defined by a straightforward manner. The elegant solution performed by the fractional heat-balance integral method has been analyzed and the main efforts have been oriented towards the evaluation of fractional (half-time) derivatives by use of approximate profile across the penetration layer.

Key words: *fractional equation, heat diffusion, half-time fractional derivative, heat-balance integral method*

Introduction

The fractional calculus [1-5] is powerful tool for solving non-linear equation with complex boundary conditions and allowing, for instance, surface flux and temperature to be known without development of the entire temperature profile in depth of the heated medium. The common method for solving fractional-order equations are purely mathematical, even tough they are approximate in nature, among them: in terms of Mittag-Leffler function [6], similarity solutions [7], Green's function [8, 9], operational calculus [10], numerical methods [11], variational iteration method [12, 13] , and differential transformations [14,15] . The present work refers to a well established method of integral solution commonly known as heat-balance integral [16]. The core of the model is the assumption of the thermal penetration layer propagating with a finite velocity. Beyond the front of this layer the medium is undisturbed. This

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idea of Goodman [17], in fact, corrects the physical incorrectness of the parabolic heat-equation where the speed of the flux is infinite.

The heat-balance integral method (HBIM), even its 50 years history, is still useful and allows many complicated problems to be solved [18-22]. The problems in the HBIM solutions are well known and may be formulates in some points, among them:

- Arbitrary of the approximate profile that is the general drawback of the classical approach conceived by Goodman [16, 17]. This disadvantage could be avoided by using methods optimizing the profile through double integrations [23-26], *i. e.* the so-called, refined integral method (RIM) [26] or by imposing thermodynamic constraints [27] and use of additional conditions derived through existing exact solutions [27-29] fractional calculus [30].
- The effect of the flux defined by the unknown profile in the right-hand side of the heat-balance integral that commonly is avoided by double integration, *i. e.* the RIM method [26].

The present article address a heat-balance integral solution of the fractional half-time model:

$$\frac{\partial^{1/2} T(x,t)}{\partial^{1/2} t} \qquad \sqrt{\alpha}\,\frac{\partial T(x,t)}{\partial x} \tag{1}$$

which has been developed by Oldham *et al.* (ref. [2], p. 200) and Agrawal [31, 32] through non-dimesionalisation, and Laplace transform of the heat-diffusion equation:

$$\frac{\partial T(x,t)}{\partial t} \quad \alpha\,\frac{\partial^2 T(x,t)}{\partial x^2} \tag{2}$$

Preliminaries

Equation (1) is second order in space and first order in time, while eq. (2) is first order in space and half order in time. Equation (1) is exact for planar geometry and a short-time approximation in its general form [ref. [2], p. 200] for cylindrical and spherical geometries [2].

Equation (1) is equivalent to the basic diffusion eq. (1) [2] and allows the heat flux and the temperature at the boundary to be determined as it was demonstrated by Agrawal [30, 31], namely:

$$q\ \ (0,t) \quad \sqrt{\lambda\rho C_{\mathrm{p}}}\,\frac{\mathrm{d}^{1/2}}{\mathrm{d}^{1/2}t}[T(0,t) \quad T_{\infty}] \tag{3a}$$

$$T(0,t) \quad \frac{1}{\sqrt{\lambda\rho C_{\mathrm{p}}}}\,\frac{\mathrm{d}^{\ \ 1/2}}{\mathrm{d}t^{\ \ 1/2}}\,q\ \ (0,t) \quad T_{\infty} \tag{3b}$$

Here $\mathrm{d}^{1/2}/\mathrm{d}t^{1/2}$ and $\mathrm{d}^{-1/2}/\mathrm{d}t^{-1/2}$ are the right-side fractional semi-derivative and integral as integro-differential operators [1-3] – see appendix A.

In accordance with the Riemann-Liouville (RL) definitions [1-4] the semi-derivatives of $T(x, t)$ with respect to the space coordinate and the time are defined as:

$$\frac{^{\mathrm{RL}}\partial^{1/2} T(x,t)}{\partial\ \ \partial^{1/2}} \quad \frac{1}{\Gamma\ \frac{1}{2}}\,\frac{\mathrm{d}}{\mathrm{d}x}\,{}_{0}^{\ x}\,\frac{T(z,t)}{\sqrt{x \quad z}}\,\mathrm{d}z \tag{4a}$$

$$^{\mathrm{RL}}D_t^{1/2} \quad \frac{\partial^{1/2} T(x,t)}{\partial\ \ t^{1/2}} \quad \frac{1}{\Gamma\ \frac{1}{2}}\,\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{0}^{\ t}\,\frac{T(z,u)}{\sqrt{t \quad u}}\,\mathrm{d}u \tag{4b}$$

Solutions of (1) and (2) describing the entire temperature distribution in a semi-infinite medium have been developed either numerically, by series [5] or other methods as mentioned above. Now, we will demonstrate how the HBIM [16, 17] employing a parabolic profile with unspecified exponent [26-28] works. The idea is explained next.

To be precise, eq. (1) is the so-called Dirac's equation [34]:

$$({}^{RL}D_t^{1/2}u)(t,x) = \lambda_D \frac{\partial u(x,t)}{\partial x}, \quad (t > 0,\ x \in R) \tag{5a}$$

$$\lim_{|x| \to \infty} u(t,x) = 0, \quad ({}^{RL}I_t^{1/2}u)(0+,x) = \delta_D(x) \tag{5b,c}$$

with $\lambda_D < 0$; $\delta_D(x)$ is the Dirac's delta function. The fundamental solution of (5a,b,c) is [30, 31]:

$$u(x,t) = \frac{1}{\lambda_D t}\varphi\left(-\frac{1}{2}, 0; -\frac{x}{\lambda_D \sqrt{t}}\right) = -\frac{\frac{x}{t^3}}{2\lambda_D\sqrt{t}}\exp\left(-\frac{x^2}{2\lambda_D t}\right), \quad x > 0 \tag{5d}$$

That to some extend approaches the well-know exact solution of eq. (2) – see Appendix E for more details.

In term of Caputo derivatives, the initial value problem is [34]:

$$({}^{C}D_t^{\mu}u)(t,x) = \lambda_D \frac{\partial u(x,t)}{\partial x}, \quad (t > 0, x \in R, 0 < \mu < 1) \tag{6a}$$

$$\lim_{|x| \to \infty} u(t,x) = 0, \quad u(0+,x) = g(x) \tag{6b,c}$$

with a general solution:

$$u(x,t) = u_{g,\lambda_D}(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{\mu,1}(-i\lambda_D k t^{\mu})G(k)\mathrm{e}^{-ikx}\mathrm{d}k \tag{6d}$$

where $G(k)$ is the Fourier transform of the initial condition $g(x)$ and $E_{\mu,\beta}(z)$ is the bi-parametric Mittag-Leffler special function [1] . This is a *localized solution* because $\lim_{|x| \to \infty} u(t,x) = 0$. With $g(x) = \delta_D(x)$, the fundamental solution turns out to eq. (5d).

Aim

The article addresses some major points, among them:

(1) Integral-balance approach to the solution of the fractional eq. (1), termed here as fractional heat-balance integral method (FHBIM), following the idea of the classical Goodman's HBIM.
(2) Benchmark solutions of two basic cases allowing comparisons to exact solutions and estimations of errors.

Problem statement – HBIM approach

HBIM approach

The HBIM approach to the fractional equations will be demonstrated through solution of the sub-model (1) by two classical examples concerning specified temperature and defined thermal flux at the boundary $x = 0$, respectively. However, the idea will be briefly explained in general.

Let us suggest that the heat propagate up to a depth into the medium and further the temperature field is being undisturbed. At the boundaries of the thermal layer the classical conditions hold, namely:

$$x = 0, \quad T = T_s \tag{7a}$$

$$x = \delta, \quad T = T_\infty \tag{7b}$$

$$x \to \delta, \quad \frac{\partial T}{\partial x} \to 0 \tag{7c}$$

Besides, the thermal penetration depth $\delta(t)$ holds the initial condition:

$$t = 0, \quad \delta(t) = 0 \tag{7d}$$

In accordance with the Goodman concept, at any time t, the following energy balance should be satisfied:

$$\int_0^\delta \frac{\partial^{1/2}}{\partial t^{1/2}} T(x,t)\ \mathrm{d}x = \sqrt{\alpha} \int_0^\delta \frac{\partial T}{\partial x} \tag{8a}$$

that yields:

$$\int_0^\delta \frac{\partial^{1/2}}{\partial t^{1/2}} T(x,t)\ \mathrm{d}x = \sqrt{\alpha}\left(T\big|_{x=\delta} - T\big|_{x=0}\right) = \sqrt{\alpha}\,(T_s - T_\infty) \tag{8b}$$

To some extent, looking at the RHS of eq. (7b) the result is equivalent to that provided by the double integration approach in solution of the classical HBIM [23, 24], and the technique used by RIM [24-26].

Next, let us suggest that the temperature distribution across the thermal layer is approximated by an approximate function $T_a(x)$ depending only of the space co-ordinate x within $0 < x < \delta(t)$. The application of the boundary conditions (6a,b,c) yields a profile expressed as a function of x and coefficients depending on $\delta(t)$. Now, let us replace in eq. (7) $T(x, t)$ by the approximating function $T_a(x, \delta)$, namely:

$$\int_0^\delta \frac{\partial^{1/2}}{\partial t^{1/2}} T_a(x,t)\ \mathrm{d}x = \sqrt{\alpha}\,(T_s - T_\infty) \tag{9a}$$

$$I_{fr} = \int_0^\delta \frac{\partial^{1/2}}{\partial t^{1/2}} T_a(x,t)\ \mathrm{d}x \tag{9b}$$

Now, the main problem is the evaluation of the fractional heat-balance integral (9b) through particular expression of $T_a(x, \delta)$ and the definition of the half-time derivative (4b). The integral (9a) is termed hereafter – *Half-time heat-balance integral* (HT-HBI).

Some preliminary thoughts

Before starting with solutions, we have to comment that from mathematical point of view the approximate function used by the HBIM only satisfies the heat-balance integral but not the original equation [30]. That's correct, but we have to remember that HBIM stands to reason because three main issues stay behind it, among them:

(1) The starting eq. (2) is physically incorrect since it leads to infinite propagation of the flux across the layer. The Goodman approach corrects this by introduction of the thermal layer concept that physically makes the problem solution close by that provided by the correct hyperbolic equation. Briefly, the front movement existing in both the hyperbolic equation solution and the real processes is not an outcome of the Fourier's parabolic equation.

(2) The integration over the space coordinate means physical an energy balance over the thermal layer (only within it there is a heat transfer) at a given time t.

(3) The choice of the approximating function is a matter of argument and it is affected by the type of boundary conditions at $x = 0$ [28, 29]. However, we have to remember, that at a small increment of the time t we have almost stationary heat transfer bounded by $0 < x < \delta$ and almost all known approximations used in HBIM (see the comprehensive analysis in [29]) quite well approach the static temperature profile. Hence, we with the integration over the space co-ordinate (the heat balance integral) we, practically, froze the time and show that temperature profile expands along x as the thermal depth grows with time. All this profiles are similar and satisfy the energy balance of the solid. This mechanistic explanation quite well gives details of the physics of the phenomena modeled by (1) and (2). Moreover, the fractional diffusion equation referring sub-diffusion problems [34] the heat (mass) propagation (diffusion) is so slow that the concept of the penetration layer becomes essential in view of the fact that it really exists [35, 36].

Therefore, there is challenge to test how the HBIM works with a fractional time derivative and what is the technology of such a solution. Furthermore, what is the outcome as a profile and predicted surface temperature/flux with respect to the solution provided by the basic model (2)? This is a good option to perform a HBIM solution of a fractional equation prior to apply the same technology to the fractional heat-wave equation. Last but not least, the browsing of the literature and available solutions of fractional diffusion equations reveals that this is, in fact, the first attempt to apply the HBIM methodology to this branch of models. Many answers to the questions and ideas raised are provided by the next benchmark solutions.

Benchmark exercises and analyses

Two basic examples with simplest boundary conditions at the medium from surface are solved by the method suggested that allows wide area for comparison of the results developed to those provided by either approximate or exact solution of the basic model (2). The program of the benchmarks solution is it follows:

– definition of the approximate profile,
– fractional heat-balance integral,
– thermal penetration depth, and
– calibration of the exponent of the approximate profile.

Approximate temperature profile

Several cases will be exemplified by expressing $T_a(x, \delta)$ as a parabolic profile with unspecified exponent [26-28] and various boundary conditions at $x = 0$

$$T_a(x, t) = a + b(1 + cx)^n \tag{10a}$$

or in dimensionless form:

$$\Theta_\mathrm{a}(x,t) = \frac{T - T_4}{T_\mathrm{s} - T_\infty} = (1 - cx)^n \tag{10b}$$

as will be demonstrated further in this work.

Any another approximate profile such as polynomial [16, 19, 29] or exponential [39, 40] can be used, but the parabolic one (10a,b) allows to demonstrate two basic issues of the method developed in this work :

– how the fractional defined boundary conditions at $x = 0$ allows to calibrate the profile exponent, and
– to demonstrate in an explicit manner the evaluation of the fractional heat-balance integral.

Example 1. Fixed temperature at x = 0

This classical problem allows to compare the exact solution of eq. (2), its HBIM solution and that of the sub-model eq. (1) through the FHBIM. Applying the boundary conditions (6a,b,c) to eq. (10) we get:

$$a = T_\infty, \quad b = T_\mathrm{s} - T_\infty, \quad c = \frac{1}{\delta} \tag{11a,b,c}$$

that yields:

$$T_\mathrm{a}(x,t) = T_\infty + (T_\mathrm{s} - T_\infty)\left(1 - \frac{x}{\delta}\right)^n \tag{12}$$

The exponent is still unspecified and its exact value will be discussed further in this work.

Example 2. Specified flux at x = 0, q"(0, t) = q₀

Let the approximate profile (10) satisfies the following boundary and initial conditions within the thermal layer $0 < x < \delta$, namely:

$$x = 0, \quad -\lambda\frac{\partial T}{\partial x} = q''(0,t) = q_0 = \text{const.} \tag{13a}$$

$$x = \delta, \quad T = T_\infty \tag{13b}$$

$$x = \delta, \quad \frac{\partial T}{\partial x} = 0 \tag{13c}$$

and

$$t = 0, \quad \delta(t) = 0 \tag{14}$$

with the conditions (25a,b,c) the coefficients of the approximate profiles are [27]:

$$a = T_\infty, \quad b = q_0\frac{\delta}{\lambda n}, \quad c = \frac{1}{\delta} \tag{15a,b,c}$$

Hence,

$$T_\mathrm{a}(x,t) = T_\infty + q_0\frac{\delta}{\lambda n}\left(1 - \frac{x}{\delta}\right)^n \tag{16a}$$

In dimensionless forms as:

$$\Theta_\mathrm{a} = \frac{T_\mathrm{a}(x,t) - T_\infty}{T_\mathrm{s} - T_\infty} = \frac{q_0}{T_\mathrm{s} - T_\infty}\frac{\delta}{\lambda n}\left(1 - \frac{x}{\delta}\right)^n \quad \text{or} \quad \Theta_\mathrm{a} = \frac{T_\mathrm{a}(x,t) - T_\infty}{q_0\dfrac{\delta}{\lambda}} = \frac{1}{n}\left(1 - \frac{x}{\delta}\right)^n \tag{16b,c}$$

The next step is the evaluation of the time semi-derivative (4b) with substitution of $T(x, t)$ by $T_a(x, t)$ as it is done in the next section.

Fractional heat-balance integral

This section addresses the integration of eq. (9b) containing a half-time derivative (RL) over the thermal penetration layer, namely:

$$D_t^{1/2} = \frac{\partial^{1/2} T(x,t)}{\partial t^{1/2}} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{T(x,u)}{\sqrt{t-u}} \mathrm{d}u = \frac{1}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{T(x,u)}{\sqrt{t-u}} \mathrm{d}u \tag{17}$$

with $\Gamma(1/2) = \pi^{1/2}$ [2, 3].

The substitution of eq. (17b) into eq. (9b) yields:

$$I_{\mathrm{fr}} = \int_0^\delta \left[\frac{1}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{T(x,u)}{\sqrt{t-u}} \mathrm{d}u \right] \mathrm{d}x = \frac{1}{\sqrt{\pi}} \int_0^\delta \left[\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{T(x,u)}{\sqrt{t-u}} \mathrm{d}u \right] \mathrm{d}x \tag{18a}$$

Denoting:

$$\Phi(x,t) = \int_0^t \frac{T(x,u)}{\sqrt{t-u}} \mathrm{d}u \tag{18b}$$

we get

$$I_{\mathrm{fr}} = \int_0^\delta \left[\frac{1}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{T(x,u)}{\sqrt{t-u}} \mathrm{d}u \right] \mathrm{d}x = \frac{1}{\sqrt{\pi}} \int_0^\delta \left[\frac{\mathrm{d}}{\mathrm{d}t} \Phi(x,t) \right] \mathrm{d}x \tag{18c}$$

Applying the Leibniz rule to eq. (18c) we have:

$$I_{\mathrm{fr}} = \frac{1}{\sqrt{\pi}} \int_0^\delta \left[\frac{\mathrm{d}}{\mathrm{d}t} \Phi(x,t) \right] \mathrm{d}x = \frac{1}{\sqrt{\pi}} \left[\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\delta \Phi(x,t) \mathrm{d}x - \Phi(\delta,t) \frac{\mathrm{d}\delta}{\mathrm{d}t} \right] \tag{19a,b}$$

Let denote:

$$I_{\mathrm{fr}} = \frac{1}{\sqrt{\pi}} (I_{\mathrm{A}} - I_{\mathrm{B}}) \tag{19c}$$

where

$$I_{\mathrm{A}} = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^\delta \Phi(x,t) \mathrm{d}x \quad \text{and} \quad I_{\mathrm{B}} = \Phi(\delta,t) \frac{\mathrm{d}\delta}{\mathrm{d}t} \tag{19d,e}$$

If the exact profile $T(x, t)$ is replaced by the approximate one $T_a(x, t)$, then the $\Phi(x, t)$ defined by eq. (18b) is approximated as:

$$\Phi_{\mathrm{a}}(x,t) = \int_0^t \frac{T_{\mathrm{a}}(x,u)}{\sqrt{t-u}} \mathrm{d}u \tag{20}$$

Hence, we get a way to find the equation defining the time evolution of the thermal penetration depth $\delta(t)$.

FHBI to Example 1

Now, let us develop the integrals at issue by replacing $T(x, t)$ by $\Theta_a(x, \delta)$ (*Example 1*), namely:

$$\frac{T - T_\infty}{T_{\mathrm{s}} - T_\infty} = \Theta_{\mathrm{a}} = \left(1 - \frac{x}{\delta}\right)^n \tag{21a}$$

Hence, the function $\Phi(x, t)$ is replaced by $\Phi_{\Theta\text{-a}}(x, t)$ expressed as:

$$\Phi_{\Theta\text{-a}}(x,t) = \int_0^t \Theta_{\mathrm{a}}(x,u)\frac{1}{\sqrt{t-u}}\mathrm{d}u \tag{21b}$$

Prior to the next step of the integration, we stress the attention on the fact *that at a given time moment t*, $\delta(t)$ *is a fixed length*. The later is quite important in the integration performed next (see 22b,c). If this statement seems strange, let us recall that if the length of the slab is L and the pre-heating period is over, and then $\delta(t)$ is replaced by L in the profile. Now, let us integrate step-by step, with eq. (21b) using as a benchmark the case of prescribed temperature at $x = 0$ (*Example 1*):

$$I_A^{\mathrm{T}} = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta \Phi_{\mathrm{a}}(x,t)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta\int_0^t \frac{\Phi_{\Theta\text{-a}}(x,\delta)}{\sqrt{t-u}}\mathrm{d}u\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta\int_0^t \left(1-\frac{x}{\delta}\right)^{\mathrm{n}}\frac{1}{\sqrt{t-u}}\mathrm{d}u\mathrm{d}x \tag{22a}$$

We integrate in the rectangle $0 \le x \le \delta$, $0 \le u \le t$ and the change in the order of integration in eq. (22a) yields:

$$I_A^{\mathrm{T}} = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{d}u\int_0^\delta \frac{T_{\mathrm{a}}(x,\delta)}{\sqrt{t-u}}\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{d}u\int_0^\delta \frac{\left(1-\dfrac{x}{\delta}\right)^n}{\sqrt{t-u}}\mathrm{d}x = \frac{d}{\mathrm{d}t}\int_0^t \frac{\delta}{n+1}\frac{1}{\sqrt{t-u}}\mathrm{d}u \tag{22b}$$

Further

$$I_A^T = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta}{n+1}\int_0^t \frac{1}{\sqrt{t-u}}\mathrm{d}u = \frac{\mathrm{d}}{\mathrm{d}t}\left(2\sqrt{t}\frac{\delta}{n+1}\right) \tag{22c}$$

Next, since $\Phi_{\Theta\text{-a}}(\delta, t)= 0$, the second term of eq. (19b) becomes zero, namely:

$$\Phi_{\Theta\text{-a}}(\delta,t)\frac{\mathrm{d}\delta}{\mathrm{d}t} = 0 \tag{23}$$

Hence

$$I_{\mathrm{fr}}^{\mathrm{T}} = \frac{1}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t}\left(2\sqrt{t}\frac{\delta}{n+1}\right) = \frac{2}{\sqrt{\pi}}\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\sqrt{t}) \tag{24a}$$

that is

$$I_{\mathrm{fr}}^{\mathrm{T}} = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta \left(\frac{\partial^{1/2}}{\partial t^{1/2}}\Theta_{\mathrm{a}}\right)\mathrm{d}x = \frac{2}{\sqrt{\pi}}\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\sqrt{t}) \tag{24b}$$

FHBI to Example 2

Following the technology in evaluations of the evaluations of the fractional heat-balance integral we have:

$$I_{\mathrm{fr}}^{\mathrm{q}} = \frac{1}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta\int_0^t \frac{q_0}{T_{\mathrm{s}}-T_\infty}\frac{\delta}{\lambda n}\left(1-\frac{x}{\delta}\right)^n\frac{1}{\sqrt{t-u}}\mathrm{d}u\mathrm{d}x \tag{25a}$$

because at $x = \delta$

$$\frac{q_0}{T_{\mathrm{s}}-T_\infty}\frac{\delta}{\lambda n}\left(1-\frac{x}{\delta}\right)^n\frac{\mathrm{d}\delta}{\mathrm{d}t} = 0 \tag{25b}$$

From eq. (25a) we have:

$$I_{\mathrm{fr}}^{\mathrm{q}} = \frac{q_0}{\lambda(T_{\mathrm{s}} - T_\infty)}\frac{1}{n}\frac{1}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\delta}\int_0^{t}\delta\left(1-\frac{x}{\delta}\right)^n \frac{1}{\sqrt{t-u}}\,\mathrm{d}u\,\mathrm{d}x \tag{26a}$$

Because the approximated function is limited within the range of integration, then following the Fubini's theorem by changing the order if integration in eq. (26a) we get:

$$I_{\mathrm{fr}}^{\mathrm{q}} = \frac{q_0}{\lambda(T_{\mathrm{s}} - T_\infty)}\frac{1}{n}\frac{1}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{t}\mathrm{d}u\int_0^{\delta}\delta\left(1-\frac{x}{\delta}\right)^n \frac{1}{\sqrt{t-u}}\,\mathrm{d}x = \int_0^{t}\frac{\delta^2}{n+1}\frac{1}{\sqrt{t-u}}\,\mathrm{d}u \tag{26b}$$

That leads to:

$$I_{\mathrm{fr}}^{\mathrm{q}} = \int_0^{\delta}\frac{\partial^{1/2}}{\partial t^{1/2}}\Theta_{\mathrm{a}}\,\mathrm{d}x = \frac{q_0}{\lambda(T_{\mathrm{s}} - T_\infty)}\frac{1}{n(n+1)}\frac{2}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t}(\delta^2\sqrt{t}) \tag{26c}$$

Thermal penetration depth

The penetration depth depends on the FHBI and the flux at $x = 0$ upon the boundary conditions imposed.

With Example 1 (*prescribed temperature*) we have:

$$I_{\mathrm{fr}}^{\mathrm{T}} = \int_0^{\delta}\frac{\partial^{1/2}}{\partial t^{1/2}}\Theta_{\mathrm{a}}\,\mathrm{d}x = \frac{2}{\sqrt{\pi}}\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\sqrt{t}) = \sqrt{\alpha}\int_0^{\delta}\frac{\partial\Theta}{\partial x}\mathrm{d}x \tag{27a}$$

that yields

$$\frac{2}{\sqrt{\pi}}\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\sqrt{t}) = \sqrt{\alpha} \tag{27b}$$

With Example 2 (*prescribed flux*) we have:

$$I_{\mathrm{fr}}^{\mathrm{q}} = \int_0^{\delta}\frac{\partial^{1/2}}{\partial t^{1/2}}\Theta_{\mathrm{a}}\,\mathrm{d}x = \frac{q_0}{\lambda(T_{\mathrm{s}} - T_\infty)}\frac{1}{n(n+1)}\frac{2}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t}(\delta^2\sqrt{t}) = \sqrt{\alpha}\int_0^{\delta}\frac{\partial\Theta}{\partial x}\mathrm{d}x \tag{28a}$$

and

$$\frac{1}{n(n+1)}\frac{2}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t}(\delta^2\sqrt{t}) = \sqrt{\alpha}\frac{\delta}{n} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\delta^2\sqrt{t}) - 2(n+1)\sqrt{\pi}\sqrt{\alpha}\delta = 0 \tag{28b}$$

The solutions of eqs. (27b) and (28b) are:

Prescribed temperature

Equation (27b) evolves to:

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} + \frac{\delta}{2t} = \frac{1}{\sqrt{t}}\sqrt{\alpha}\sqrt{\pi}\frac{n+1}{2} \tag{29a}$$

with a solution (by Maple 7)

$$\delta_T(t) = D\sqrt{t} + \frac{C_1}{\sqrt{t}}, \quad D = \sqrt{\alpha}\sqrt{\pi}\frac{n+1}{2} \tag{29b}$$

Prescribed flux

Equation (28b) evolves to:

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} \quad \frac{\delta}{4t} \quad \frac{1}{\sqrt{t}}\frac{(n \quad 1)\sqrt{\pi}\sqrt{\alpha}}{\sqrt{t}} \tag{30a}$$

Equation (32b) has a solution (by Maple 7)

$$\delta_{\mathrm{q}}(t) \quad \frac{4}{3}B\sqrt{t} \quad \frac{C_2}{\sqrt[4]{t}}, \quad B \quad (n \quad 1)\sqrt{\pi}\sqrt{\alpha} \tag{30b}$$

In both cases, eqs. (29a) and (30b), the initial conditions are: $\delta(0) = \delta_{\mathrm{T}}(0) = \delta_{\mathrm{q}}(0) = 0$. Hence, we have the only reasonable $C_1 = 0$ and $C_2 = 0$ that finally defines the thermal layer depths, namely:

$$\delta_{\mathrm{T}}(t) \quad \sqrt{\alpha t}\sqrt{\pi}\,\frac{n \quad 1}{2} \tag{31a}$$

$$\delta_{\mathrm{q}}(t) \quad \sqrt{\alpha t}\,\frac{4}{3}\sqrt{\pi}(n \quad 1) \tag{31b}$$

The choice of $C_1 = 0$ and $C_2 = 0$ comes from physical reasons since eqs. (29a) and (30b) give $\delta \quad \infty$ at $t = 0$, that is equivalent to infinite propagation of the temperature field at the onset of the diffusion process.

Calibration of the exponents of the approximate profiles

In accordance with the results developed in the previous section (eqs. 31a,b) we obtained the following approximate profiles:

Example 1

$$T_{\mathrm{a}}(x,t) \quad T_{\infty} \quad (T_{\mathrm{s}} \quad T_{\infty}) \quad 1 \quad \left(\frac{x}{\sqrt{\alpha t}\,\dfrac{\sqrt{\pi}(n \quad 1)}{2}}\right)^{n} \tag{32a}$$

Example 2

$$T_{\mathrm{a}}(x,t) \quad T_{\infty} \quad \frac{q_0\delta}{\lambda}\frac{\delta}{n} \quad 1 \quad \left(\frac{x}{\sqrt{\alpha t}\,\dfrac{4}{3}\sqrt{\pi}(n \quad 1)}\right)^{n} \tag{32b}$$

With the classic Goodman method applied to eq. (2) the accuracy of the approximated solution depends on the choice of the exponent of the parabolic profile, commonly taken as $n = 2$ or $n = 3$. However, the arbitrary choice of the exponent n raises a question about its refining with respect to minimization of errors in the approximate solution. It was demonstrated in [27, 28], that when the solution of eq. (2) by the HBIM was developed the correct determination of the exponent needed one boundary condition more in addition to BC (5a,b,c) imposed by the thermal layer concept of Goodman. More precisely, this additional boundary condition means the heat flux to be known when the surface temperature is specified. With respect to the solution of

(2), this approach was developed by either the exact solution [27, 28] of Carslaw *et al.* [37] – *termed here as "exact solution" of eq. (2)* – or by fractionally defined flux and temperature, as presented by eqs. (3a,b). The second approach, for example, allows the HBIM to be applied to quite complex boundary conditions.

With regard to the solution of eq. (1), under the conditions of Example 1, by the parabolic profile (10), the additional boundary condition at $x = 0$ is the surface flux. This implies that both solutions, the approximate and the exact one, have to provide equal thermal fluxes at the boundary. Therefore, the flux at is immediately defined from the half-time derivative of the surface temperature, that is the exact solution is given by eq . (3a) and with both profile the following condition has to hold:

$$q\ (0,t) \quad q_{\mathrm{fr}}(0,t) \quad q_{\mathrm{a}}(0,t) \tag{33a}$$

Hence, irrespective of function representing the solution, exact or approximate, the heat flux has to be equal if we have to obey the process physics. Where the fluxes provided by both solutions are:

$$q_{\mathrm{fr}}(0,t) \quad \sqrt{\lambda\rho C_{\mathrm{p}}}\,\frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}[T(0,t) \quad T_{\infty}] \quad \text{and} \quad q_{a}(0,t) \quad \lambda(T_{\mathrm{s}} \quad T_{\infty})\frac{n}{\sqrt{\alpha t}\,\dfrac{\sqrt{\pi}(n \quad 1)}{2}} \tag{33b,c}$$

Equating the boundary heat fluxes provided by both solutions we have – $T(0, t) = T_{\mathrm{s}} =$ = const.:

$$\sqrt{\lambda\rho C_{\mathrm{p}}}\,\frac{1}{\sqrt{\pi}\sqrt{t}} \quad \sqrt{\lambda\rho C_{\mathrm{p}}}\,\frac{1}{\sqrt{\pi}\sqrt{t}}\,\frac{2}{n \quad 1} \tag{34}$$

and

$$\frac{\sqrt{\pi}}{\sqrt{\pi}} \quad \frac{2n}{n \quad 1} \qquad n \quad 1 \tag{35}$$

Hence, the approximate profile is:

$$T_{\mathrm{a}}(x,t) \quad (T_{\mathrm{s}} \quad T_{\infty})\ 1 \quad \frac{x}{\sqrt{\pi}\sqrt{\alpha t}} \qquad T_{\mathrm{a}}(x,t) \quad (T_{\mathrm{s}} \quad T_{\infty})\ 1 \quad \frac{x}{1.772\sqrt{\alpha t}} \tag{36}$$

With the prescribed flux problem (*Example 2*) , since the flux is defined at $x = 0$, the following condition has to hold:

$$T_{\mathrm{a}}(0,\ t) = T(0,\ t) \tag{37a}$$

Here $T_{\mathrm{s}}(0, t)$ is defined by eq. (3b), setting eq. (1) at $x = 0$, because $(\ T/\ x)_{x=0}$ defines the surface flux.

Hence

$$T_{\mathrm{fr}}(x \quad 0) \quad \frac{1}{\sqrt{\lambda\rho C_{\mathrm{p}}}}\,\frac{\mathrm{d}\ ^{1/2}q_0}{\mathrm{d}t\ ^{1/2}} \quad T_{\infty} \quad \frac{1}{\sqrt{\lambda\rho C_{\mathrm{p}}}} 2q_0\sqrt{\frac{t}{\pi}} \quad T_{\infty} \tag{37b}$$

Because the semi-integral on the l. h. s. of eq. (3b) is $(d^{-1/2}C)/(\mathrm{d}t^{-1/2}) = 2C(t/\pi)^{1/2}$ [2] where C is any constant. From eq. (37b) with the HBIM profile defined by eq. (32b) making equal the temperatures at $x = 0$ we have:

$$\frac{1}{\sqrt{\lambda\rho C_\mathrm{p}}}2q_0\sqrt{\frac{t}{\pi}} \quad \frac{q_0}{\lambda}\frac{\delta}{n} \quad \frac{1}{\sqrt{\lambda\rho C_\mathrm{p}}}2q_0\sqrt{\frac{t}{\pi}} \quad \frac{1}{\sqrt{\lambda\rho C_\mathrm{p}}}2q_0\sqrt{\pi}\,\frac{4}{3n^2}\sqrt{t} \tag{38}$$

This equation defines n as:

$$3n^2 \quad 2\pi \quad n \quad \sqrt{\frac{2\pi}{3}} \quad \sqrt{2.094395} \quad 1.447 \tag{39}$$

Brief comments on FHBI and the solutions developed thereof

The general idea was applied and verified by two simple examples with simplest boundary conditions well-known in the literature. The improvement of the method of the HBIM in the cases with the half-time derivative sub-model is that the surface temperature or the flux can be derived directly by eqs. (3a) and (3b) – these outcomes of the direct setting of eq. (1) at $x = 0$. This option, in fact, allows closing the set of equations needed to find all parameters of the profile (10) as it was analyzed in [28]. The classical Goodman method provides only 3 conditions (5a,b,c) that work correctly with the profile (10) – see the discussion in [28]. The determination of the exponent n needs an additional condition to be imposed on the profile. These are general comments, which summarize previous notions and remarks but in general with the half-time derivative sub-model the deficiency in the boundary conditions are not obvious unlike the case when the same method and profile are applied to eq. (2) (see [27, 29]).

To this end it is better to know what are the expressions of δ provided by the integral solution of the sub-model (1), that of the basic one (2) as well to compare results provided by them to those coming from exact solutions. As in the classical studies on HBIM we compare the flux and temperatures at $x = 0$ since, in fact, the values of the exponents were calibrated at this point. The exactness and correctness of this approach will be discussed further in this work.

A principle point in the development of the fractional HBI is the integration developed. In Appendices B and C an alternative approach is provided, more physical rather than mathematical, giving results exactly the same as those developed by eqs. (24b) and (28b). In addition, integrations by Maple 7 of both examples at issue is given in Appendix D; giving the same results as limits when $x \quad \delta(t)$.

Numerical tests with the approximate solutions

The following numerical experiments test the accuracy of the profiles developed by the methodology developed in Example 1 and Example 2 and calibration of the exponent at $x = 0$.

These numerical tests address two major issues:
– prediction of the boundary values (flux or temperature) by approximate profiles, and
– prediction the temperature profile across the thermal penetration layer.

Boundary values

Test to Example 1 – thermal penetration depth

With the fixed temperature at $x = 0$ we have:

Sub-model (1)
$$\delta_1^T \quad \sqrt{\alpha t}\,\frac{\sqrt{\pi}(n \quad 1)}{2} \tag{40a}$$

Basic model (2) $$\delta_2^T = \sqrt{\alpha t}\sqrt{2n(n+1)} \qquad (40b)$$

Test to Example 2 – thermal penetration depth

With fixed flux at $x = 0$ the results are:

Sub-model (1) $$\delta_1^q = \delta(t) = \sqrt{\alpha t}\,\frac{4\sqrt{\pi}}{3n} \qquad (41b)$$

Basic model (2) $$\delta_2^q = \sqrt{\alpha t}\sqrt{n(n+1)} \qquad (41b)$$

The subscripts 1 and 2 refer to the sub-model (1) and the basic model (2), respectively. The superscripts T and q denotes the boundary condition at $x = 0$. The results about the HBIM solution the model (2) are taken from [27, 28].

Accuracy of the solution of $x = 0$

(A) *Example 1* (*fixed temperature boundary condition*) – Prediction of the boundary flux

With the developed profile we have:

– Sub-model (1) $$q_{a1}(0,t) = (T_s - T_\infty)\frac{2n}{\sqrt{\alpha t}\sqrt{\pi}(n+1)} \Rightarrow q_{a1}(0,t) \approx 0.564(T_s - T_\infty)\frac{\lambda}{\sqrt{\alpha t}},\ n = 1 \qquad (42a)$$

– Model (2) $$q_{a2}(0,t) = (T_s - T_\infty)\frac{1}{\sqrt{\alpha t}\sqrt{2n(n+1)}} = 0.322(T_s - T_\infty)\frac{\lambda}{\sqrt{\alpha t}},\quad n = 1.75 \qquad (42b)$$

– Exact solution of (2) $$q_e(0,t) = 0.318(T_s - T_\infty)\frac{\lambda}{\sqrt{\alpha t}} \qquad (42c)$$

– Fractional relationship, eq. (3a) $$q_{fr}(0,t) = (T_s - T_\infty)\frac{1}{\sqrt{\pi}\sqrt{\alpha t}} = 0.564(T_s - T_\infty)\frac{\lambda}{\sqrt{\alpha t}} \qquad (42d)$$

Hence, the HBIM solution of the sub-model (1) gives a result, eq. (42a), which overestimates the exact solution, eq. (42c), of the basic model. In fact, the approximate profile of the sub-model (1) was calibrated through the fractional boundary flux:

$$q_{fr}(0,t) = \sqrt{\lambda\rho C_p}\,\frac{d^{1/2}}{dt^{1/2}}[T(0,t) - T_\infty] = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\alpha t}}(T_s - T_\infty) \approx 0.564 \qquad (43a)$$

If the profile (32a) is calibrated at $x = 0$ through the exact solution of the basic model (2), then from:

$$q_e(0,t) = 0.318(T_s - T_\infty)\frac{\lambda}{\sqrt{\alpha t}} \quad \text{and} \quad q_a(0,t) = \lambda(T_s - T_\infty)\frac{n}{\sqrt{\alpha t}\,\dfrac{\sqrt{\pi}(n+1)}{2}} \qquad (43b,c)$$

we have

$$0.318\frac{1}{\sqrt{\alpha t}} = \frac{2n}{\sqrt{\alpha t}\sqrt{\pi}(n+1)} \Rightarrow \frac{n}{n+1} = 0.281 \Rightarrow n = 0.390 \qquad (43d)$$

The result of eq. (43d) gives:

$$q_{a1}(0,t) = (T_s - T_\infty)\lambda \frac{2n}{\sqrt{\alpha t}\sqrt{\pi}(n+1)} = 0.898(T_s - T_\infty)\frac{\lambda}{\sqrt{\alpha t}}, \quad n = 0.390 \tag{43e}$$

which is incorrect since we need $n = 1$. The more correct result is provided by calibrating through the fractional boundary flux, *i. e.* with $n = 1$ because, the relationship (3a) is more general as a solution then the well known exact one.

(B) *Example 1* (*fixed flux boundary condition*) – Prediction of the boundary temperature

– Sub-model (1)
$$T_{a1}(0,t) = T_\infty + \frac{4\sqrt{\pi}}{3n^2}\frac{q_0}{\lambda}\sqrt{\alpha t} = T_\infty + 1.128\frac{q_0}{\lambda}\sqrt{\alpha t}, \quad n = 1.447 \tag{44a}$$

– Model (2)
$$T_{a2} = T_\infty + \sqrt{\alpha t}\frac{q_0}{\lambda n}\sqrt{n(n+1)} = T_\infty + 1.128\frac{q_0}{\lambda}\sqrt{\alpha t}, \quad n = 3.65 \tag{44b}$$

– Exact solution
$$T_e = T_\infty + 2\frac{q_0}{\lambda\sqrt{\pi}}\sqrt{\alpha t} = T_\infty + 1.128\frac{q_0}{\lambda}\sqrt{\alpha t} \tag{44c}$$

– Fractional relationship, eq. (3b)
$$T_{fr}(0,t) = T_\infty + 1.128\frac{q_0}{\lambda}\sqrt{\alpha t} \tag{44d}$$

In this case all the three solutions provide practically equal surface temperatures and the discrepancies emerging in the comparison of the results provided by Example 1 do not exist.

Temperature profiles

The temperature profile established through the approximate parabolic profile can be expressed through the similarity variable $\eta = x/(\alpha t)^{1/2}$. This gives a possibility to compare the approximate solution with those assumed as exact ones. Hence, as functions of η we have:

Example 1

$$\Theta_{aT} = \frac{T - T_\infty}{T_s - T_\infty} = \left[1 - \eta\frac{2}{\sqrt{\pi}(n+1)}\right]^n = (1 - 0.564\eta), \quad n = 1 \tag{45a}$$

At the same time, the exact solution of the basic problem (2), corresponding to Example 1 is:

$$\Theta_e = 1 - \operatorname{erf}\left(\frac{\eta}{2}\right) \tag{45b}$$

Moreover, the HBIM solution of the basic problem (2) with the same parabolic profile is [27]:

$$T_a(x,t) = T_\infty + (T_s - T_\infty)\left[1 - \frac{x}{\sqrt{\alpha t}\sqrt{2n(n+1)}}\right]^n \quad \text{with} \quad n = 1.75 \tag{45c}$$

Figures 1(a,b) show the temperature profiles developed by the approximate solutions together with the exact solution of the basic problem (2). The plots clearly indicate that up to $\eta \approx 0.5$ with $n = 1$, eq. (45a) all the solutions coincide with negligible errors. This upper limit, reveals, that

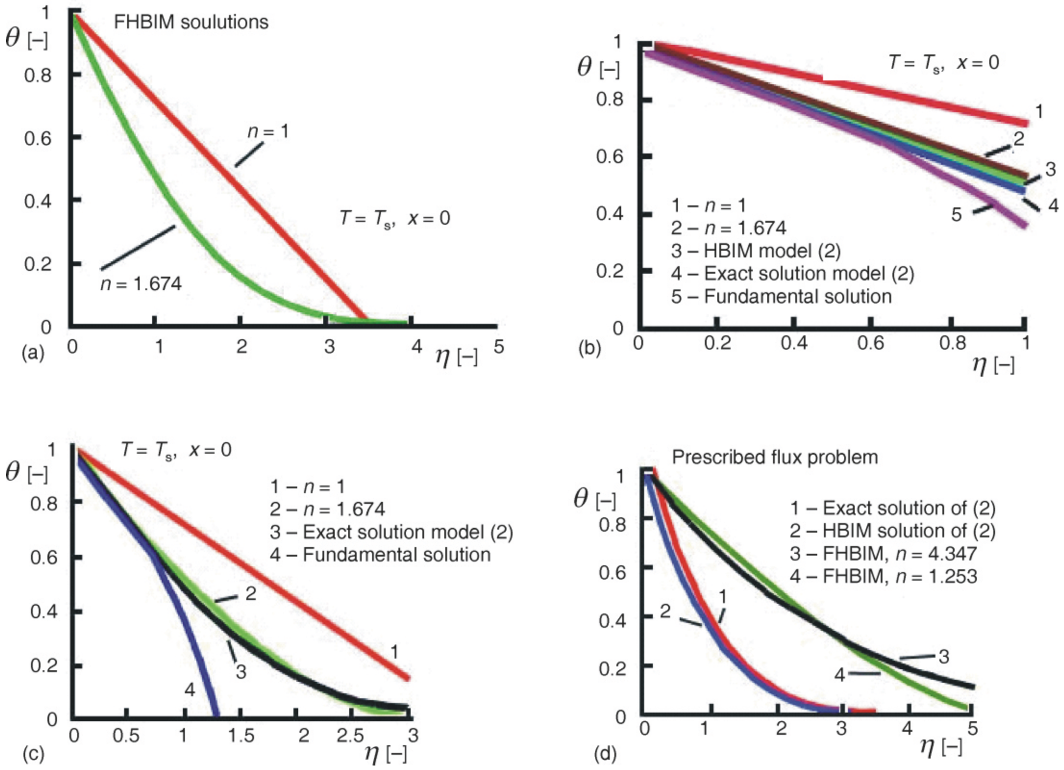

Figure 1. Temperature profiles – numerical results developed by different equations developed by both HBIM and FHBI. The fundamental solution in (b) and (c) is that described by (5d)

with $\alpha \sim 10^{-5}$ W/m^{2}K and $t \sim 1$ s we have $x \approx 5 \cdot 10^4$ m, for example, that is quite large as distance. On other hand, with large-time approximation, with $0 \le \eta \le 0.5$ we have satisfactory solutions in a short distance beyond the point $x = 0$. The increase in the range where the integral approach outcomes will approach the exact solution needs different approach in determination of the exponent n of the approximate parabolic profile. This problem will be discussed in the next section.

Example 2

$$T_{\mathrm{a}}(x,t) = T_\infty + \frac{q_0}{\lambda}\frac{\delta}{n}\left[1 - \frac{x}{\sqrt{\alpha t}\,\frac{4}{3}\sqrt{\pi}(n-1)}\right]^n = \Theta_{\mathrm{aq}}$$

$$\frac{T - T_\infty}{\frac{q_0}{\lambda}\frac{\delta}{n}} = (1 - 0.292\eta)^{1.447}, \quad n = 1.447 \tag{46a}$$

The profiles are shown in fig. 1(d). The discrepancy with respect to the exact solution becomes evident with increase in the value of η. The improvement of the profile will be discussed in the next section concerning the global minimization of the error of approximate solution.

Least-square approach in defining the optimal exponent

Preliminaries

The Langford's criterion [38] concerning the solution of eq. (2) with the classical HBIM is:

$$E(t) = \int_0^{\delta(t)} \left[\frac{\partial T_e(x,t)}{\partial x^2} - \frac{1}{\alpha} \frac{\partial T_e(x,t)}{\partial t} \right]^2 \mathrm{d}x \approx 0 \tag{47}$$

If the approximate profile is a good enough solution of the classical heat-conduction equation, then the following function $E(t)$ should reach its minimum; with the exact solution we have $E(t) = 0$. The Langford's criterion, in fact is a minimization of the error of the eq. (2) that does not talk about the optimal value of n. In this context, the Myers' method [26] applied to eq. (2) is a straightforward minimization procedure starting from eq. (47) and providing the optimal value of the exponent of the profile (10). Here, we will use the idea of eq. (47) formulating an error function $E_{1/2}(t)$ in the form:

$$E_{1/2}(t) = \int_0^{\delta(t)} \left[\frac{\partial^{1/2}}{\partial t^{1/2}} T_a(x,t) + \sqrt{\alpha} \frac{\partial T_a}{\partial x} \right]^2 \mathrm{d}x \approx 0 \tag{48a}$$

The optimal n should minimize $E_{1/2}(t)$ and provide the desired approximate profile. Hence, by use approximate profile in eq. (48a) we have:

$$F_{1/2}(t) = \int_0^{\delta(t)} \frac{\partial^{1/2}}{\partial t^{1/2}} T_a(x,t) + \sqrt{\alpha} \frac{\partial T_a}{\partial x} \quad \text{and} \quad E_{1/2}(t) = \int_0^{\delta(t)} [F_{1/2}(x,t)]^2 \mathrm{d}x \tag{48b,c}$$

The criterion is a general condition requiring the approximating function to satisfy the domain equation.

We will demonstrate how this approach works with respect to the optimal values of the exponents leading to a global minimization of $E_{1/2}(t)$, in both examples analyzed here.

Numerical test

Example 1

With Example 1, we have:

$$\Theta_a^T = \frac{T_a(x,t) - T_\infty}{T_s - T_\infty} = \left(1 - \frac{x}{\delta}\right)^n \tag{48d}$$

and the terms of $F_{1/2}(x, t)$ are:

$$\frac{\partial^{1/2}}{\partial t^{1/2}} \Theta_a(x,t) = \frac{1}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}t} \left[2\sqrt{t} \left(1 - \frac{x}{\delta}\right)^n \right] = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{t}} \left(1 - \frac{x}{\delta}\right)^n + \frac{2}{\sqrt{\pi}} \sqrt{t} \frac{x}{\delta^2} n \left(1 - \frac{x}{\delta}\right)^{n-1} \frac{\mathrm{d}\delta}{\mathrm{d}t} \tag{49a}$$

$$\frac{\partial \Theta_a}{\partial x} = -\frac{nx}{\delta} \left(1 - \frac{x}{\delta}\right)^{n-1} \tag{49b}$$

Hence

$$F_{1/2}(x,t) = \frac{1}{\pi} \frac{1}{\sqrt{t}} \left(1 - \frac{x}{\delta}\right)^n + \frac{2}{\pi} \sqrt{t} \frac{x}{\delta^2} \frac{\mathrm{d}\delta}{\mathrm{d}t} - \frac{nx}{\delta} \left(1 - \frac{x}{\delta}\right)^{n-1} \tag{50a}$$

then

$$E_{1/2}(t)_{\mathrm{T}} = \int_0^{\delta}\left[\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{t}}\left(1-\frac{x}{\delta}\right)^{n} - \frac{2}{\pi}\sqrt{t}\,\frac{x}{\delta^2}n\left(1-\frac{x}{\delta}\right)^{n-1}\frac{\mathrm{d}\delta}{\mathrm{d}t} - \frac{nx}{\delta}\left(1-\frac{x}{\delta}\right)^{n-1}\right]^2\mathrm{d}x \quad (50\mathrm{b})$$

Using the expressions: $\delta = \delta_1^{\mathrm{T}} = (\alpha t)^{1/2}[\pi(n+1)]/2$ and $\mathrm{d}\delta/\mathrm{d}t = \mathrm{d}\delta_1^{\mathrm{T}}/\mathrm{d}t = [(\alpha t)^{1/2}/4t^{1/2}]\pi^{1/2}\cdot(n+1)$, and following the form of the expressions developed in [30], the error function is represented as:

$$E_{1/2}(t)_{\mathrm{T}} = \frac{\sqrt{\alpha}}{\sqrt{t}}e_{1/2}^{\mathrm{T}}(n,t) \quad (50\mathrm{c})$$

where

$$e_{1/2}^{\mathrm{T}}(n,t) = \frac{1}{2\sqrt{\pi}(2n+1)} - \frac{2(n+1)}{3\sqrt{\pi}} + \frac{\sqrt{t}}{\sqrt{\alpha}}\frac{4}{\pi(n+1)(n+2)} + t\frac{n\sqrt{\pi}(n+1)}{2(2n-1)(2n+1)} + \frac{1}{2(2n-1)} - 2\sqrt{t} \quad (50\mathrm{d})$$

Since the term $\alpha^{1/2}/t^{1/2}$ decreases in time, the minimization of $E_{1/2}(t)_{\mathrm{T}}$ depends on the second term $e_{1/2}^{\mathrm{T}}(n,t)$ [26]. Following the Myers's idea [26] the minimization of $e_{1/2}^{\mathrm{T}}(n,t)$ should be performed at $t = 0$, that yields:

$$e_{1/2}^{\mathrm{T}}(n,0) = \frac{1}{2\sqrt{\pi}(2n+1)} - \frac{2(n+1)}{3\sqrt{\pi}} + \frac{1}{2(2n-1)} \Rightarrow \min. \quad (51)$$

The function (51) has a complex character but we look for a non-negative value of $n > 1$ through minimization of $e_{1/2}^{\mathrm{T}}(n,0)$. The only non-negative root of $(\mathrm{d}/\mathrm{d}n)(e_{1/2}^{\mathrm{T}}(n,0)) = 0$ that is greater than is $n = 1.6745$ and min. $e_{1/2}^{\mathrm{T}}(n,0) = 3.82$. This value of n differs from that established in Example 1, *i. e.* $n = 1$, through calibration at $x = 0$. To this end, we have to recall that the differences in the values of n come from the principles applied to find it: (1) $n = 1$ was established in Example 1 through an operation making equal the boundary flux provided by the approximate solution to that calculated by the fractional half-time derivative, without any influence of the temperature profile in depth the medium; (2) $n = 1.6745$ was established taking into account the approximate temperature profile and minimizing the error $E_{1/2}(t)_{\mathrm{T}}$, that, in fact, is a more general condition. Hence, with the fixed temperature at $x = 0$ we have to take $n = 1.6745$ and the profile becomes – see fig. 1(c).

$$T_{\mathrm{a}}(x,t) = T_{\infty} + (T_{\mathrm{s}} - T_{\infty})\left(1 - \frac{x}{2.369\sqrt{\alpha t}}\right)^{1.6745} \quad (52\mathrm{a})$$

With this new profile and repeating the calculations expressed by (39a,b,c) we have:

– Sub-model (1) $q_{\mathrm{a1}}(0,t) = (T_{\mathrm{s}} - T_{\infty})\dfrac{1}{\pi\sqrt{\alpha t}} = 0.422(T_{\mathrm{s}} - T_{\infty})\dfrac{1}{\sqrt{\alpha t}}, \quad n = 1.6745$ (52b,c)

– Exact solution of (2) $q_e(0,t) = 0.318(T_{\mathrm{s}} - T_{\infty})\dfrac{1}{\sqrt{\alpha t}}$ (52d)

Thus, with $n = 1.6745$ the sub-model (1) overestimates the boundary heat flux with about 30.2% that is inacceptable. Obviously, if we try to balance between the approach used in Example 1, providing exact values of the surface flux, and the error minimization through the

Myer's approach to the least-squares method, then, the optimal value of n is between 1 and n = 1.6745. We have to remember, that in Example 1 only physical assumptions and correct mathematical manipulations were applied, while in the Langford approach and the minimization procedure thereof we apply only mathematical tools no matter what stay behind the mathematical operators. In this, context, remember that the optimal $n = 1.6745$ was determined by setting $t = 0$, that makes the function $e^{\mathrm{T}}_{1/2}(n,0)$ dependent only on n. However, this is a mathematical trick only because at $t = 0$ there is no heat diffusion in the body, the temperature profile does not exist, and the valid results are only those attached to the surface $x = 0$, *i. e.* the surface temperature T_s and the heat flux defined by eq. (3a); this directly address the approach used in Example 1 and n = 1 as outcome. The approach with $t = 0$, neglects the time-dependent terms of the error function that decrease in time and focus the efforts at the minimization procedure at the "stationary terms" depending on the exponent n.

Figure 1(c) shows plots of all solutions together with the fundamental one expressed by eq. (5d). All approximate profiles match the profile (5d) within $0 \quad \eta \quad 0.5$.

Example 2

With the profile (27d) taking into account that $\delta \quad \delta_1^q \quad [(\alpha t)^{1/2}/5]\pi^{1/2}(n+1)$ and $\mathrm{d}\delta/\mathrm{d}t =$ $= \mathrm{d}\delta_1^q/\mathrm{d}t = (\alpha^{1/2}/10t^{1/2})\pi^{1/2}(n+1)$ we get $E_{1/2}(t)_\mathrm{q} = (\alpha^{1/2}/t^{1/2})e^q_{1/2}(n,t)$. Then, solving $e^q_{1/2}(n,t) \quad 0$ (all operations performed by Maple 9.5) and setting $t = 0$ in the final expression we have:

$$\alpha(n \quad 1)^3 \; 0.0628 \quad \frac{30.959}{n(2n \quad 1)(2n \quad 3)} \quad 0 \tag{53}$$

The first 3 roots ($n_1 = n_2 = n_3 = -1$) are unrealistic. Further, the analytical solution performed by the Maple's operation "*solve*" provides: $n_4 = -3.173828995 - \mathrm{i}\cdot 4.274839820$, $n_5 =$ $= -3.173828995 + \mathrm{i}\cdot 4.274839820$, and $n_6 = 4.347$. Only the last root $n_6 = 4.347$ is real and non-negative. It is greater than $n = 1.447$, see eq. (39) established directly by the FBIM. With $n = 1.447$ we get $e^q_{1/2}(n,0) = 6.177809649$ while with $n = 4.347$ the error function is $e^q_{1/2}(n,0) =$ $= 0.001569$. Therefore, with $n = 4.347$, the new profile is:

$$T_\mathrm{a}(x,t) \quad T_\infty \quad 0.772\frac{q_0}{\lambda}\sqrt{\alpha t} \; 1 \quad \frac{x}{3.358\sqrt{\alpha t}}^{4.347} \tag{54}$$

Calculations, similar to those performed with (40a,b,c), give:

Sub-model (1) $T_{\mathrm{a}1}(0,t) \quad T_\infty \quad \pi\frac{q_0}{5\lambda n}(n \quad 1)\sqrt{\alpha t} \qquad T_{\mathrm{a}1}(0,t) \quad T_\infty 0.772\frac{q_0}{\lambda}\sqrt{\alpha t}, \; n \quad 4.347$ (55a)

Exact solution $T_\mathrm{e} \quad T_\infty \quad 2\frac{q_0}{\lambda\sqrt{\pi}}\sqrt{\alpha t} \qquad T_\infty \quad 1.128\frac{q_0}{\lambda}\sqrt{\alpha t}$ (55b)

Therefore, the exponent assuring a minimum mean-squares error – see fig. 1(d), of the fractional sub-equation underestimates the surface temperature with about 31% that of the exact solution.

Brief on the optimal profile

The general lesson of the least-square tests in estimation of the optimal exponent is:

(1) When the surface temperature (flux) has to be defined, then the general approach developed here through calibrating the profile at $x = 0$, has to be applied.

(2) When the temperature profile has to be modeled and the overall error over the entire thermal layer has to be minimized then, the least-squares approach is the additional tool leading to the optimal value of the exponent n. However, this step minimizes the global error but gives inacceptable boundary fluxes and temperatures, as it was demonstrated.

Conclusions

In this paper the general framework of an integral solution to a diffusion equation with half-time fractional derivative was presented. The fractional sub-model of the classical heat-diffusion equation was especially chosen because both equations are equivalent in plane geometry. The main approach in the classical integral approach (an approximating function dependent only on the space co-ordinate) gives an advantage in evaluation of the fractional half-time derivative represented by a convolution integral as it was demonstrated by *Example 1* and *Example* 2 . This approach simplifies the calculations and, in fact, we evaluate the convolution integral of the approximate function. The next step involving the integration over the space co-ordinate is almost the same as in the classical HBIM. The outcome of this new step in the approximate solution of fractional equations is that it provides solutions almost the same as those when the basic solutions are solved. The test eq. (1) was especially chosen as mentioned above, since it has an integer analogue (2) that allows easily comparing the results and elucidating the emerging problems, among them:

The FHBIM applied to the fractional sub-model provides almost the same expressions about the heat penetration depth and practically equal results about the surface temperature and flux (see the numerical tests (42a,b,c,d) and (44a,b,c,d) as those provided by the HBIM and exact solutions of the integer model (2).

The fractional sub-model provides an artificial boundary condition $x = 0$ allowing the number of the equations to be equal to the number of the unknown parameters of the parabolic profile. This artificial condition comes immediately after setting both sides of the sub-model at $x = 0$. This is an advantage, since in the case of the integer model (2) this condition does not exist [26-28].

The optimal exponent established through the complete set of boundary conditions of FHBIM, is lower that that derived through minimization of the global mean-square error of the sub-model over the entire penetration depth. This result is not strange since similar problem exist in the HBIM solution of integer model [22, 26, 29].

The FHBIM uses an elegant technology that has some advantages with respect to the direct determination of the exponent of the profile through the mean-square error minimization. The method gives lowest global error of approximation but the consequent calculations of the surface temperature (or flux) by the exponent provided by it is unacceptably overestimated (or underestimated).

The method demonstrated in this work is, in fact, the first attempt to solve a fractional equation by integral method. The HBIM is well-known and widely applicable but never tested with fractional-time diffusion equation. The technology developed in this work allows developing solutions even with more complex boundary conditions then those used in this work; this problem is beyond the scope of the present article but still unsolved.

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Nomenclature

a – coefficient in the prescribed temperature, [K]
b – coefficient in the prescribed temperature profile, [K]
c – coefficient in the prescribed temperature profile, [m^{-1}]
$E(t)$ – defined integral as a measure of the error of approximations (eq. 47)
$E_{1/2}(t)$ – global (integral) error function defined by eq. (48a), [–]
$E_{1/2}(t)_{\mathrm{T}}$ – global (integral) error function defined by eq. (50b) for the fixed temperature problem, [–]
$E_{1/2}(t)_{\mathrm{q}}$ – global (integral) error function defined by eq. (53) for the fixed flux problem, [–]
$e^{\mathrm{T}}_{1/2}(n,t)$ – error function defined by the integral $E_{1/2}(t)_{\mathrm{T}}$, [–]
$e^{\mathrm{q}}_{1/2}(n,t)$ – error function defined by the integral $E_{1/2}(t)_{\mathrm{q}}$, [–]
$F_{1/2}$ – function defined by (48b) and the integral $E_{1/2}(t)_{\mathrm{T}}$ (48c), [–]
n – exponent in the prescribed temperature profile, [–]
q , q_{s} – surface heat flux, [Wm^{-2}]
q_{a} – surface heat flux provided by the approximate temperature profile, [Wm^{-2}]
q_{e} – surface heat flux provided by the exact temperature profile, [Wm^{-2}]
q_{fr} – surface heat flux provided by the fractional (half-time), [Wm^{-2}]
T – temperature, [K]
T_{a} – temperature defined by the approximate solution, [K]
T_{e} – temperature defined by the exact solution, [K]
T_{s} – surface temperature (commonly at $x = 0$, see the context), [K]
T_∞ – temperature of the undisturbed medium, [K]
t – time, [s]
u – dummy variable in the fractional-time derivative (see eq. 4b)
x, y – co-ordinates (x is used in some contexts as general independent variable), [m]
z – dummy variable in the fractional-space derivative (see eq. 4a)

Greek letters

α – thermal diffusivity, [m^{2}s^{-1}]
δ – thermal layer depth, [m]
λ – thermal conductivity, [Wm^{-1}K]
μ – fractional orders of the derivatives, [–]
$\Theta^{\mathrm{T}}_{\mathrm{a}}$ – dimensionless temperature [$= (T - T_\infty)/(T_{\mathrm{s}} - T_\infty)$], see eq. (42d)

Superscripts

C – Caputo derivative
RL – Riemann-Liouville (Example 1) erivative
q – prescribed flux problem (Example 2)
T – prescribed temperature problem (Example 1)

Subscripts

a – approximate
e – exact
fr – fractional defined

Abbreviations

HBIM – heat-balance integral method
FHBIM– fractional heat-balance integral method

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APPENDICES

Appendix A

Fractional derivative – definition and properties

Fractional calculus is the branch of calculus that generalizes the derivative of a function to non-integer order, allowing calculations such as deriving a function to 1/2 order. Despite "generalized" would be a better option, the name "fractional" is used for denoting this kind of derivative. The Riemann-Liouville derivative is the most used generalization of the derivative. It is based on Cauchy's formula for calculating iterated integrals:

$$^{RL}D^{\mu} f(x) = \frac{1}{\Gamma(1-\mu)} \int_0^x \frac{f(t)}{(x-t)^{\mu-1}} \mathrm{d}t \qquad \text{(A-1)}$$

Fractional derivatives satisfy quite well all the properties that one could expect from them, despite some of them are only characteristic of integer order differentiation and some other have restrictions. Assuming further $\mu = 1/2$ for seek of clarity of the explanation in the main text we have:

– linearity
$$D^{\mu}[af(x) + bg(x)] = aD^{\mu}f(x) + bD^{\mu}g(x) \qquad \text{(A-2)}$$

– composition rule $D^{\mu} D^{\gamma} f(x) = \dfrac{\mathrm{d}^{\mu+\gamma}}{\mathrm{d}x^{\mu+\gamma}} f(x)$

– with μ = 1/2 (a semi-derivative) used in the work

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}} C = \frac{C x^{-1/2}}{\sqrt{\pi}} \tag{A-3}$$

$$D^{1/2} f(Cx) = C^{1/2} \frac{\mathrm{d}^{1/2} f(Cx)}{\mathrm{d}(Cx)^{1/2}} \tag{A-4}$$

where C is any constant

Appendix B

Alternative approach in development of the fractional heat-balance integral – Example 1

The time semi-derivative (4b) with substitution of $T(x, t)$ by $T_a(x, t)$

$$T_a(x,t) = T_\infty + (T_s - T_\infty)\left(1 - \frac{x}{\delta}\right)^n \tag{B-1}$$

is

$$D_{ta}^{1/2} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \left[T_\infty + (T_s - T_\infty)\left(1 - \frac{x}{\delta}\right)^n\right] \frac{1}{\sqrt{t-u}} \mathrm{d}u \tag{B-2}$$

where $D_{ta}^{1/2}$ denotes a time semi-derivative calculated through the approximating profile $T_a(x, t)$ with $\Gamma(1/2) = \pi^{1/2}$ [2, 3].

In fact $\delta = \delta(t)$, but at a given time t the profile depends only on the space-coordinate x. This assumption allows, considering the function $T_a(x, \delta)$ *as time-independent* and moving it outside the convolution integral in eq. (13), to get:

$$D_{ta}^{1/2} = \frac{1}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}t} \left[T_a(x,\delta) \int_0^t \frac{1}{\sqrt{t-u}} \mathrm{d}u\right] = \frac{1}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}t} \left[T_a(x,\delta)\left(-2\sqrt{t-u}\right)\Big|_0^t\right] = \frac{2}{\sqrt{\pi}} T_a(x,\delta)\sqrt{t} \tag{B-3}$$

From (B-4) we have:

$$D_{ta}^{1/2} = \frac{1}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}t} \left\{2\sqrt{t}\left[T_\infty + (T_s - T_\infty)\left(1 - \frac{x}{\delta}\right)^n\right]\right\} \tag{B-4}$$

Now, the integral of (B-3) becomes:

$$\int_0^\delta \frac{2}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}t} \left\{\sqrt{t}\left[T_\infty + (T_s - T_\infty)\left(1 - \frac{x}{\delta}\right)^n\right]\right\} \mathrm{d}x = \sqrt{\alpha} \int_0^\delta \frac{\partial T}{\partial x} \tag{B-5}$$

Now, applying the Leibniz rule for differentiation under the integral sign (the classical step in the Goodman method) we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\delta}\frac{2}{\sqrt{\pi}}\sqrt{t}\;T_\infty \quad (T_\mathrm{s} \quad T_\infty)\;1 \quad \frac{x}{\delta}^{\;n}\mathrm{d}x \quad \frac{2}{\pi}T_\infty\delta\sqrt{t} \quad \sqrt{\alpha}(T_\mathrm{s} \quad T_\infty) \qquad \text{(B-6)}$$

From (B-6) we get:

$$\frac{2}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\sqrt{t}}{n \quad 1} \quad \sqrt{\alpha} \quad \frac{\mathrm{d}\delta}{\mathrm{d}t}\sqrt{t} \quad \delta\frac{1}{2t} \quad \frac{1}{\sqrt{t}}\sqrt{\alpha}\sqrt{\pi}\frac{n \quad 1}{2} \qquad \text{(B-7)}$$

Integrating (B-7) we have:

$$\delta(t) \quad D\sqrt{t} \quad \frac{C_1}{\sqrt{t}}, \quad D \quad \sqrt{\alpha}\sqrt{\pi}\frac{n \quad 1}{2} \qquad \text{(B-8)}$$

Obeying the initial condition we have the only reasonable $C_1 = 0$ that finally defines the thermal layer depth:

$$\delta(t) \quad D\sqrt{t} \quad \sqrt{\alpha t}\sqrt{\pi}\frac{n \quad 1}{2} \qquad \text{(B-9)}$$

The choice of $C_1 = 0$ comes from physical reasons since eq. (29a) gives $\delta \quad \infty$ at $t = 0$, that is equivalent to infinite propagation of the temperature field at the onset of the diffusion process. The result is exactly the same as that developed through other technique of integration – see eq. (29b).

Besides, in ref. [5 – p. 89], it is especially mentioned that the function defining the motion of the front in Stefan-like problems (in the present case this is $\delta(t)$ should be considered *as a constant*, independent of the time t when the $D_\mathrm{t}^{1/2}$(RL) has to be calculated. In fact, this is the approach demonstrated by (B-3)-(B-5).

Appendix C

Alternative approach in development of the fractional heat-balance integral – Example 2

With the approximate profile in the FHBI we have:

$$T_\mathrm{a}(x,t) \quad T_\infty \quad q_0\frac{\delta}{\lambda n} \quad 1 \quad \frac{x}{\delta}^{\;n} \qquad \text{(C-1)}$$

Integrating (C-1) from 0 to δ, see eq. (B-3), we have:

$$\int_0^{\delta}\frac{\partial^{1/2}}{\partial t^{1/2}} \; T_\infty \quad q_0\frac{\delta}{\lambda n} \quad 1 \quad \frac{x}{\delta}^{\;n}\mathrm{d}x \quad \sqrt{\alpha}\int_0^{\delta}\frac{\partial T}{\partial x}\mathrm{d}x \qquad \text{(C-2)}$$

Applying the same technology as to eq. (B-3) we have:

$$D_\mathrm{ta}^{1/2} \quad \frac{1}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t} \quad 2\sqrt{t}\;T_\infty \quad q_0\frac{\delta}{\lambda n} \quad 1 \quad \frac{x}{\delta}^{\;n} \qquad \text{(C-3)}$$

Now the integration with respect to the space co-ordinate x yields:

$$\int_0^{\delta}\frac{2}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t} \quad \sqrt{t}\;T_\infty \quad q_0\frac{\delta}{\lambda n} \quad 1 \quad \frac{x}{\delta}^{\;n} \quad \mathrm{d}x \quad \sqrt{\alpha}\int_0^{\delta}\frac{\partial T}{\partial x}\mathrm{d}x \qquad \text{(C-4)}$$

Replacing the space derivative in the r. h. s. of eq. (C-4) by $T/ \ x = -(\alpha^{1/2}q_0/\lambda)[1 - (x/\delta)^{n-1}]$ and applying the Leibniz rule we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\delta} \frac{2}{\sqrt{\pi}}\sqrt{t}\ T_\infty \quad q_0\frac{\delta}{\lambda n}\ 1 \quad \frac{x}{\delta}^{\ n}\ \mathrm{d}x \quad \frac{2}{\sqrt{\pi}}T_\infty\delta\sqrt{t} \quad \frac{\sqrt{\alpha}q_0}{\lambda}\int_0^{\delta} 1 \quad \frac{x}{\delta}^{\ n\ 1}\mathrm{d}x \tag{C-5}$$

The r. h. s. of eq. (C-5) is:

$$\frac{\sqrt{\alpha}q_0}{\lambda}\int_0^{\delta} 1 \quad \frac{x}{\delta}^{\ n\ 1}\mathrm{d}x \quad \frac{1}{n}\frac{\sqrt{\alpha}q_0}{\lambda}\delta$$

Then integrating eq. (C-5) from 0 to δ, see eqs. (B-6 and B-7), we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta^2\sqrt{t}) \quad \frac{\sqrt{\pi}}{2n}\frac{\sqrt{\alpha}}{\lambda}\delta \quad 0 \tag{C-6}$$

Equation (C-7), evolves to:

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} \quad \frac{\delta}{4t} \quad \frac{\sqrt{\pi}}{2n}\frac{\sqrt{\alpha}}{\sqrt{t}} \tag{C-7}$$

Equation (C-7) has a solution:

$$\delta(t) \quad \frac{16}{3}B\sqrt{t} \quad \frac{C_2}{\sqrt[4]{t}}, \quad B \quad \frac{\sqrt{\pi}\sqrt{\alpha}}{4n} \tag{C-8a,b}$$

With eq. (C-8a) the initial condition $\delta(0) = 0$ the only reasonable value of the integration constant is $C_2 = 0$. Hence, the thermal layer depth is:

$$\delta(t) \quad \sqrt{\pi}\sqrt{\alpha t}\,\frac{4}{3n} \tag{C-9}$$

The results is exactly the same as eq. (31b).

Appendix D

Integration by Maple 7 of the FHBI

$$\int_0^{\delta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{t}\frac{T_\mathrm{a}(x,u)}{\sqrt{t \quad u}}\mathrm{d}u \ \ \mathrm{d}x \tag{D-1}$$

– Example 1

The Maple program line

```
> int(int(((1-x/y(t))^n/sqrt(t-u)), u=0..t), x=0..y(t));
```

provides the solution:

$$\lim_{x \quad \delta(t)} \ 2\frac{\delta(t)\sqrt{t}}{n \quad 1} \quad \frac{\delta(t) \quad x}{\delta(t)}^{\ n\ 1} \quad 1 \quad 2\frac{\delta(t)^2\sqrt{t}}{n \quad 1} \tag{D-2}$$

– Example 2

The Maple program line

```
int(int((y(t)*(1-x/y(t))^n/sqrt(t-u)), u=0..t), x=0..y(t));
```

provides the solution:

$$\lim_{x \quad \delta(t)} \quad 2\frac{\delta(t)^2\sqrt{t}}{n \quad 1} \quad \frac{\delta(t) \quad x}{\delta(t)}^{n \quad 1} \quad 1 \quad 2\frac{\delta(t)^2\sqrt{t}}{n \quad 1} \tag{D-3}$$

Appendix E

Dirac-like fractional equation

Equation (1) has been studied by Kilbas *et al*. [32], Pierantozzi *et al*. [33], and Usero *et al*. [34] from the standpoint of an initial value problem involving the Dirac's fractional equation:

$$({}^{\mathrm{RL}}D_t^{1/2}u)(t,x) \quad \lambda_D \frac{\partial u(x,t)}{\partial x}, \quad (t \quad 0, x \quad R) \tag{E-1}$$

$$\lim_{|x| \quad \infty} u(t,x) \quad 0, \quad ({}^{\mathrm{RL}}I_t^{1/2}, u)(0 \quad ,x) \quad \delta_D(x) \tag{E-2}$$

where $\delta_D(x)$ is the Dirac's delta function.

With $\lambda_D < 0$ (as in the case at issue, see eq. (1), with $-\alpha^{1/2}$), the fundamental solution is [32]:

$$u(x,t) \quad \frac{1}{\lambda_D t}\varphi \quad \frac{1}{2}, 0; \frac{x}{\lambda_D\sqrt{t}} \quad \frac{\frac{x}{\sqrt{t^3}}}{2\lambda_D\sqrt{t}}\exp\frac{x^2}{2\lambda_D t}, \quad x \quad 0 \tag{E-3}$$

The moments of this fundamental solution are given by – recall, the integration over the space co-ordinate in the method used here defines the 1st moment of the l. h. s. of eq. (1):

$$\int_{\infty}^{\infty} x^n u(x,t)\mathrm{d}x \quad (\quad \lambda_D)^n \frac{\Gamma(n \quad 1)}{\Gamma\left(\frac{n \quad 1}{2}\right)} t^{\frac{n \quad 1}{2}} \quad (n \quad 0, 1, 2, \ldots) \tag{E-4}$$

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