APPLICATION OF THE HEAT-BALANCE AND REFINED INTEGRAL METHODS TO THE KORTEWEG-DE VRIES EQUATION

by

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In this paper we consider approximate travelling wave solutions to the Korteweg-de Vries equation. The heat-balance integral method is first applied to the problem, using two different quartic approximating functions, and then the refined integral method is investigated. We examine two types of solution, chosen by matching the wave speed to that of the exact solution and by imposing the same area. The first set of solutions is generally better with an error that is fixed in time. The second set of solutions has an error that grows with time. This is shown to be due to slight discrepancies in the wave speed.

Key words: *Korteweg-de Vries, heat-balance integral method, refined integral method, travelling wave*

Introduction

As its name suggests, the heat-balance integral method (HBIM) was developed to solve problems in heat flow [1]. It has made a particular impact in the analysis of Stefan problems, where few analytical solutions exist. Since the heat equation is ubiquitous, the method has application to numerous scenarios. For example, the heat equation, diffusion equation, and porous media equation are identical, so results for heat flow carry through to diffusion and porous media problems. The Schrödinger equation is simply the heat equation with a complex diffusivity. The heat equation is used in probability and describes random walks. For this reason it is also applied in financial mathematics and is a particular limit of the famous Black-Scholes equation. It is important in Riemannian geometry and thus topology. For a discussion of applications of the heat equation see [2-4], for example. In viscous flow the heat equation is retrieved in the analysis of an impulsively moved plate in a semi-infinite viscous fluid. In fact the HBIM is an adaptation of the Karman-Pohlhausen integral method [5] for analysing boundary layers in fluid flow, see [6] for a translated account of this work.

In this paper we extend the applicability of the method to a completely different problem, namely the solution of the Korteweg-de Vries equation. We focus on the well-known travelling wave solution. Our work is inspired by a paper of Kutluay *et al*. [7]. However, this work has two basic errors (which we discuss in the section *The HBIM solution*) that lead to an incorrect representation of the wave form, so we do not follow their analysis. For our analysis we will use two methods: the standard HBIM and a refined integral method (RIM), see [8, 9]. As always we encounter the issue of the choice of approximating function. Goodman primarily employed a quadratic [1]. However, even for this simple choice Wood [10] shows six different formulations

and demonstrates that Goodmans choice is typically third best. Kutluay *et al.* [7] opt for the standard quadratic. A single cubic profile was employed in Myers *et al.* [11] when studying the melting of a sub-cooled finite block. Their choice was motivated by analysing the melting of a material initially at its solidus. Both the small argument expansion of the exact solution and an asymptotic solution lead to a cubic with no quadratic term. Antic and Hill [12] use two cubics to describe the temperature in grain and the surrounding air in a model of thermal diffusion in a grain store. Mitchell *et al.* [13] employ a quartic in a study of ablation. This choice is motivated through an analysis of the heating up stage before ablation commences. Their results are compared with an analysis of Braga *et al.* [14, 15] who use functions of the form:

$$u = a_0 + a_1(\delta - x)^n \tag{1}$$

where *n* is a non-integer, chosen so that the melting time predicted by the HBIM solution agrees with an exact analytical solution. In the current paper we employ a quartic, since this is the lowest order capable of satisfying the boundary conditions.

Problem set-up and exact solution

Consider the Korteweg-de Vries equation of the form:

$$u_t + \varepsilon u u_x + \mu u_{xxx} = 0 \tag{2}$$

The initial condition:

$$u(x,0) \quad 3C\text{sech}^2(Ax \quad D), \quad A \quad \frac{1}{2}\sqrt{\frac{\varepsilon C}{\mu}} \tag{3}$$

permits a travelling wave solution of the form:

$$u(x, t) \quad 3C\text{sech}^2(Ax \quad \varepsilon CAt \quad D) \tag{4}$$

Specifically this is a soliton moving in the positive x-direction with speed εC. We will use this solution to verify our approximate solutions.

The HBIM solution

Kutluay *et al.* [7] analyse the above problem on a fixed domain $x \quad [0, 2]$ (and therefore the parameters and initial condition must be chosen so that the wave is well contained within this region). Furthermore, the solution is restricted in time so that the wave does not approach the boundary. They then look for an approximating function of the form:

$$u = a(t)(x^2 - 2x) \tag{5}$$

which satisfies $u(0, t) = u(2, t) = 0$. Integration of eq. (2) with respect to x over the fixed domain gives:

$$\int_0^2 u_t \mathrm{d}x \quad \varepsilon \frac{u^2}{2}\bigg|_{x \ 0}^{x \ 2} \quad \mu u_{xx}\bigg|_{x \ 0}^{x \ 2} \quad 0 \tag{6}$$

Since $u = 0$ at either boundary and u_{xx} is constant this tells us that a must be constant. Kutluay *et al.* incorrectly apply the condition on u_{xx} and find $a = \alpha e^{-3\mu t/2}$, for some unknown constant α. Their error is compounded by an incorrect application of the initial condition to determine α, which transforms the quadratic form (5) to:

$$u(x, t) = 3Ce^{3\mu t/2}\text{sech}^2(Ax + D) \tag{7}$$

Obviously this is simply the initial condition with an exponential multiplier. It is therefore no surprise that their solutions appear accurate for small times. However, this quickly deteriorates and by $t = 0.5$ the errors are as high as 45%. Based on this they state the HBIM is only suitable as a very small time approximation to the solution of eq. (2) and (3).

Given that the solution is defined by a travelling wave we would expect the errors (from an appropriate approximation) to be independent of time. We therefore approach the problem in a rather different manner. Firstly, we work over a moving domain and so introduce a quantity $\delta(t)$, equivalent to the heat penetration depth in standard HBIM solutions [1]. In this case $\delta(t)$ defines the leading edge of the wave, where we assume $u(\delta, t) = 0 = u_x(\delta, t)$. To ensure smoothness we also set $u_{xx}(\delta, t) = 0$. Our approximation must track the wave peak and so we introduce the position $x_m(t)$, which defines where u is a maximum:

$$u(x_m, t) = 3C, \qquad u_x(x_m, t) = 0 \tag{8}$$

Therefore, our travelling wave is defined for $x \in [2x_m - \delta, \delta]$. For the wave to retain its form we require $\delta - x_m = w$ to be constant, where w is the half-width of the wave.

We have five boundary conditions and therefore look for a quartic approximating function. This takes the form:

$$u = 3C\left[4\frac{(\delta - x)^3}{(\delta - x_m)^3} - 3\frac{(\delta - x)^4}{(\delta - x_m)^4} \right] \tag{9}$$

which satisfies conditions (8) and $u = u_x = u_{xx} = 0$ at $x = \delta$. Note that this form only applies to the region $x_m < x < \delta$; the solution for $2x_m - \delta < x < x_m$ is obtained by reflecting in the line $x = x_m(t)$. The approximation involves the unknown $\delta(t)$. We determine δ by integrating eq. (2) with respect to x over the region $[x_m, \delta]$

$$\frac{d}{dt}\int_{x_m}^{\delta} u\,dx - \frac{d\delta}{dt} u(\delta, t) + \frac{dx_m}{dt} u(x_m, t) + \left[\frac{\varepsilon}{2} u^2 - \mu u_{xx}\right]_{x_m}^{\delta} = 0 \tag{10}$$

Using the boundary conditions and the fact that $\delta - x_m$ is constant, this reduces to:

$$\frac{d}{dt}\int_{x_m}^{\delta} u\,dx - 3C\frac{d\delta}{dt} - \frac{9\varepsilon C^2}{2} - \mu(u_{xx}|_\delta - u_{xx}|_{x_m}) = 0 \tag{11}$$

Substitution of u from (9) into this integral expression leads to an ordinary differential equation for δ, namely:

$$\frac{d\delta}{dt} = \frac{3\varepsilon C}{2} + \frac{12\mu}{w^2} \tag{12}$$

The initial condition on δ comes from the initial shape of the wave, which has $x_m(0) = -D/A$, so $\delta(0) = -D/A + w$. Equation (12) has solution:

$$\delta = \left(\frac{3\varepsilon C}{2} + \frac{12\mu}{w^2}\right)t - \frac{D}{A} + w \tag{13}$$

and so

$$x_m = \left(\frac{3\varepsilon C}{2} + \frac{12\mu}{w^2}\right)t - \frac{D}{A} \tag{14}$$

To ensure the correct wave speed we set the term in brackets to εC, which determines $w = = (24\,\mu/\varepsilon C)^{1/2}$.

An alternative formulation arises by neglecting the condition $u_{xx} = 0$ at $x = \delta$. Instead we note:

$$\frac{\mathrm{d}}{\mathrm{d}t}u[\delta(t),t] \quad \frac{\partial u}{\partial t}\Big|_{x}\,{}_{\delta} \quad \frac{\mathrm{d}\delta}{\mathrm{d}t}\,\frac{\partial u}{\partial t}\Big|_{x}\,{}_{\delta} \quad 0 \tag{15}$$

Imposing $u_x(\delta, t) = 0$ and substituting for u_t through the original eq. (2), we find $u_{xxx}(\delta, t) = 0$. Consequently we obtain:

$$u \quad 3C \quad 2\frac{(\delta \ x)^2}{(\delta \ x_m)^2} \quad \frac{(\delta \ x)^4}{(\delta \ x_m)^4} \tag{16}$$

The expressions for δ and x_m turn out to be the same as in solutions (13) and (14) and so w is unchanged.

The refined integral method (RIM) solution

The RIM is similar to the HBIM, except a second integration is carried out on the governing equation:

$$\int_{x_m}^{\delta}\int_{x_m}^{x}\frac{\partial u}{\partial t}\mathrm{d}\xi \ \mathrm{d}x \quad \frac{\varepsilon}{2}\int_{x_m}^{\delta}(u^2 \ u^2\big|_{x_m})\mathrm{d}x \quad \mu\int_{x_m}^{\delta}\ \frac{\partial^2 u}{\partial x^2} \ \frac{\partial^2 u}{\partial x^2}\Big|_{x_m} \ \mathrm{d}x \quad 0 \tag{17}$$

Further details of the method may be found in [9, 13]. This equation may be re-written as:

$$\int_{x_m}^{\delta}\int_{x_m}^{x}\frac{\partial u}{\partial t}\mathrm{d}\xi \ \mathrm{d}x \quad \frac{\varepsilon}{2}\int_{x_m}^{\delta} u^2\mathrm{d}x \quad (\delta \ x_m) \ \frac{\varepsilon}{2}u^2 \quad \mu\frac{\partial^2 u}{\partial x^2}\bigg|_{x_m} \quad 0 \tag{18}$$

using the fact that $u_x = 0$ at $x = x_m$ and $x = \delta$. The double integral can be integrated once by parts and the dummy variable ξ replaced with x:

$$\int_{x_m}^{\delta}\int_{x_m}^{x}\frac{\partial u}{\partial t}\mathrm{d}\xi \ \mathrm{d}x \quad \delta\int_{x_m}^{\delta}\frac{\partial u}{\partial t}\mathrm{d}x \quad \int_{x_m}^{\delta} x\frac{\partial u}{\partial t}\mathrm{d}x \quad \delta\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_m}^{\delta}u\mathrm{d}x \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_m}^{\delta}xu\mathrm{d}x \quad 3Cw\frac{\mathrm{d}\delta}{\mathrm{d}t}$$

Thus eq. (18) becomes:

$$\delta\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_m}^{\delta}u\mathrm{d}x \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_m}^{\delta}xu\mathrm{d}x \quad 3Cw\frac{\mathrm{d}\delta}{\mathrm{d}t} \quad \frac{\varepsilon}{2}\int_{x_m}^{\delta}u^2\mathrm{d}x \quad \delta_0 \ \frac{\varepsilon}{2}u^2 \quad \mu\frac{\partial^2 u}{\partial x^2}\bigg|_{x_m} \quad 0 \tag{19}$$

Substituting the profile (9) into eq. (19) gives:

$$\delta \quad \frac{25\varepsilon C}{14} \quad \frac{20\mu}{w^2} \ t \quad \frac{D}{A} \ w \tag{20}$$

This is the RIM equivalent of eq. (13). Matching wave speeds gives $w = (280 \, \mu/11\varepsilon C)^{1/2}$. The profile (16) leads to:

$$\delta \quad \frac{13\varepsilon C}{7} \quad \frac{15\mu}{w^2} \ t \quad \frac{D}{A} \ w \tag{21}$$

where $w = (105\mu/6\varepsilon C)^{1/2}$.

One drawback with this method of choosing w is that it requires knowledge of the wave speed, which comes from the exact solution. An alternative way to compute w is by choosing the area under the wave to match that from the initial condition. Using this method we find $w = (5/2)A$ and $(15/7)A$ for the profiles (9) and (16), respectively (irrespective of whether we use RIM or HBIM). We will present solutions where both the wave speed and area are matched in the following section.

Results

In fig. 1 we show a comparison of the HBIM solution with the exact solution, where w is determined from the correct wave speed. The parameter values are taken from the paper of Kutluay *et al*. [7], namely $\mu = 4.84 \cdot 10^{-4}$, $\varepsilon = 1$, and $C = 0.3$. We also set the initial position by choosing $D = -6$. The solid line is the exact solution (4), the dashed line represents the approximation (9), and the dotted line is the approximation (16). These approximate solutions have been calculated over the region $[x_m, \delta]$: to obtain the full profile the solution should be reflected in the line $x = x_m(t)$. For both profiles there is reasonable correlation, however, it is clear that the best approximation is given by profile (9). The L_2 norms for $u_{exact} - u_{approx}$ are 1.34 and 3.69 when u_{approx} is calculated through (9) and (16) respectively.

Figure 1. Comparison of the exact solution, (eq. (4), solid line) and HBIM solutions for correct wave speed (a) cubic and quartic, (eq. (9), dashed line) and (b) quadratic and quartic, (eq. (16), dotted line) at $t = 1$

Figure 2 shows a comparison of the RIM solutions with the exact solution. The solution for eq. (16) shows a significant improvement, with an L_2 norm of 1.75. The accuracy of the second profile, eq. (9), has deteriorated slightly with a norm of 1.52. Note that, for both the HBIM and RIM solutions these errors will be constant for all time.

When we use the condition of matching areas the error is time dependent. Initially the L_2 norm is 1.45 and 1.9 for the two different approximating functions (9) and (16), respectively, regardless of the method used. By $t = 1$ these have changed to 2.55 and 10.4 for the HBIM and 1.15 and 4.47 for RIM. Strangely, one of the RIM errors has decreased. This comes about since part of the approximate wave lies above the exact solution, as time progresses this wave moves to the left with respect to the exact wave and so there is
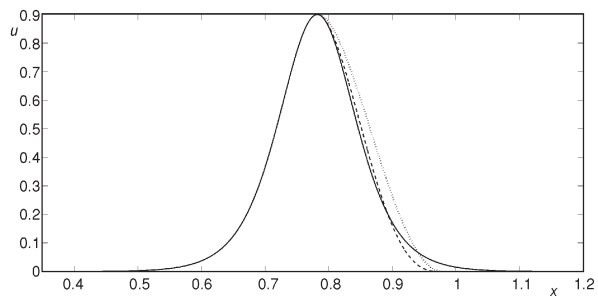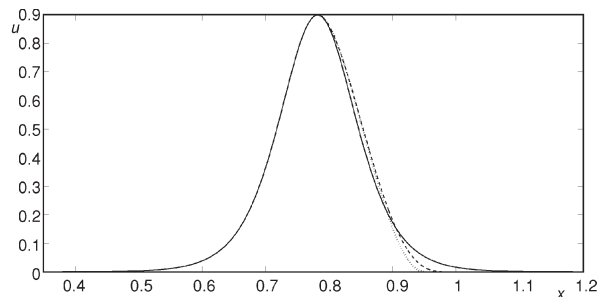

Figure 2. Comparison of the exact solution (eq. (4), solid line) and RIM solutions for correct wave speed (a) cubic and quartic, (eq. (9), dashed line) and (b) quadratic and quartic (eq. (16), dotted line) at $t = 1$

a time when a large part of the waves coincide. By $t = 3$ the errors are 5, 27 (HBIM), 2 and 10 (RIM) for the functions (9) and (16), respectively. The general growth in errors is due to an incorrect wave speed: the exact solution has a wave speed $\dot{x}_m = \varepsilon C$. If $w = 5/2A$ then eq. (14) shows:

$$\dot{x}_m \quad \frac{3\varepsilon C}{2} \quad \frac{12\mu}{w^2} \quad 3\varepsilon C \quad \frac{1}{2} \quad \frac{4}{25} \quad 1.02\varepsilon C \tag{22}$$

corresponding to a 2% error. Any error introduced by the initial approximation will therefore slowly grow as time proceeds. For eq. (16) we find $\dot{x}_m$ $0.85\varepsilon C$, an approximate -15% error, and so this wave will rapidly fall behind the exact wave form. For the RIM formulations the errors are -1% and 4%, respectively.
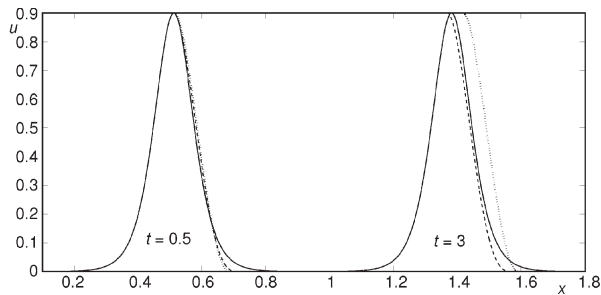
Figure 3. Comparison of the exact solution (eq. (4), solid line) and HBIM solutions for correct wave speed (a) cubic and quartic, (eq. (9), dashed line) and (b) quadratic and quartic (eq. (16), dotted line) at *t* = 0.5 and *t* = 3

The drift from the exact solution is seen clearly in fig. 3, where results are plotted at times $t = 0.5$ and 3. This only shows the RIM solutions, since these proved to be the most accurate. At $t = 0.5$ the approximations show excellent agreement with the exact solution, except in the vicinity of the $x = \delta$. By $t = 3$ the solution of eq. (16) has noticeably moved ahead of the exact solution. The solution of eq. (9) is still close but definitely deteriorating.

Conclusions

In this paper we have shown how the HBIM and RIM solutions may be applied to the Korteweg-de Vries equation. We have also used two approaches to calculate the half-width of the wave. For the former, we chose to match the wave speed of the approximate and exact solutions. The standard HBIM formulation, where the approximating function involved cubic and quartic terms, gave the smallest L_2 norm although both RIM formulations had similar values.

Of course the wave speed is not always known *a priori* and so a second method to determine the half-width was used that required matching the area under the approximate solution with that of the initial condition. In this case the RIM formulation proved best. The main drawback of this approach is that the error changes in time. Initially both HBIM and RIM formulations had the same L_2 norm, which depended on the approximating function (and with the exception of the worst HBIM solution these norms are higher than when matching the wave speed). Since the wave speeds differ slightly from the true value these errors tend to increase and so this form of solution can only be considered valid for small times, although for much longer times than suggested by Kutluay *et al.*

The approximating function involving even powers of x was chosen for two reasons. Firstly, taking the total derivative of the condition $u[\delta(t), t] = 0$ indicated the cubic term should be zero. Secondly, the small argument expansion of the exact solution involves only even powers. It is therefore surprising that the solution involving a cubic gives the best results. This highlights the difficulty in choosing an approximating function for the HBIM or RIM solutions, for which, as yet, there is no systematic method.

Nomenclature

t – temporal variable, [s] w – half-width of wave, [m]
$u(x, t)$ – wave amplitude, [m] x – spatial variable, [m]

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