THERMO ELASTIC-PLASTIC TRANSITION IN A THIN ROTATING DISC WITH INCLUSION

by

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Original scientific paper
UDC: 621.66:620.193.94
BIBLID: 0354-9836, *11* (2007), 1, 103-118

Stresses for the elastic-plastic transition and fully plastic state have been derived for a thin rotating disc with shaft at different temperatures and results have been discussed and depicted graphically. It has been observed that the rotating disc with inclusion and made of compressible material requires lesser angular speed to yield at the internal surface and higher percentage increase in angular speed to become fully plastic as compare to disc made of incompressible material. With the introduction of thermal effect the rotating disc with inclusion required lesser angular speed to yield at the internal surface. Rotating disc made of compressible material with inclusion requires higher percentage increase in angular speed to become fully-plastic as compare to disc made of incompressible material. Thermal effect also increases the values of radial and circumferential stresses at the internal surface for fully-plastic state.

Key words: *stress, displacement, rotating disc, angular speed, inclusion, temperature*

Introduction

Rotating discs are an essential part of the rotating machinery structure, *e. g*. rotors, turbines, compressors, flywheel, and computer's disc drive. The stress analysis of thin rotating discs made of isotropic material has been discussed extensively by Timoshenko and Goodier [1] in the elastic range and by Chakrabarty [2] and Heyman [3] for the plastic range. Their solutions for the problem of fully plastic state do not involve the plane stress condition, that is to say, one can obtain the same stresses and angular speed necessary for fully plastic stress of the disc without using the plane stress condition (*i. e*. $T_{zz} = 0$). Gupta and Shukla [4] obtained a different solution for the fully plastic state by using Seth's transition theory [5] and plane stress condition. This theory does not required any assumptions like an yield condition or incompressibility condition and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. It utilizes the concept of generalized strain measure and asymptotic solution at critical points or turning points of the differential equations defining the deformed field and has been successfully applied to a large number of problems [4, 7-14, 16].

Seth [6] has defined the generalized principal strain measure as:

$$e_{ii} = \int_0^{\hat{e}_{ii}} [1 - 2\hat{e}_{ii}]^{\frac{n}{2} - 1} \mathrm{d}\hat{e}_{ii} = \frac{1}{n}[1 - (1 - 2\hat{e}_{ii})^{\frac{n}{2}}], \quad (i = 1, 2, 3) \tag{1}$$

where n is the measure and $\hat{e}_{ii}$ are the Almansi finite strain components [6].

In this paper, we investigate the problem of "thermo elastic-plastic transition in a thin rotating disc with shaft" by using Seth's transition theory. Results have been discussed and depicted graphically.

Governing equations

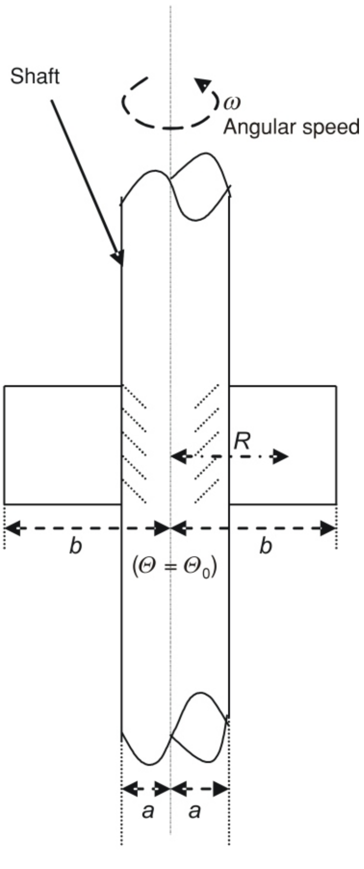

Figure 1. Geometry of rotating disc

We consider a thin annular disc with central bore of radius a and outer radius b (fig. 1). The disc, produced of material of constant density, is mounted on a rigid shaft.

The disc is rotating with angular speed ω about a central axis perpendicular to its plane. The thickness of disc is assumed to be constant and is taken sufficiently small so that the disc is effectively in a state of plane stress, that is, the axial stress T_{zz} is zero. The temperature at the central bore of the disc is Θ.

The displacement components in cylindrical polar co-ordinates are given by [6]:

$$u = r(1 - \beta), \quad v = 0, \quad w = dz \tag{2}$$

where β is position function, depending on $r = (x^2 + y^2)^{1/2}$ only, and d is a constant.

The finite strain components are given by Seth [6] as:

$$\begin{aligned} \hat{e}_{rr} &= \frac{\partial u}{\partial r} - \frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2 = \frac{1}{2}[1 - (r\beta' + \beta)^2] \\ \hat{e}_{\theta\theta} &= \frac{u}{r} - \frac{u^2}{2r^2} = \frac{1}{2}(1 - \beta^2) \\ \hat{e}_{zz} &= \frac{\partial w}{\partial z} - \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 = \frac{1}{2}[1 - (1 - d)^2] \\ \hat{e}_{r\theta} &= \hat{e}_{\theta z} = \hat{e}_{zr} = 0 \end{aligned} \tag{3}$$

where $\beta' = \mathrm{d}\beta/\mathrm{d}r$ and meaning of superscripts A is Almansi. Substituting eqs. (3) in eq. (1), the generalized components of strain are:

$$
\begin{aligned}
e_{rr} \quad & \frac{1}{n}[1 \quad (r\beta \quad \beta)^n] \\
e_{\theta\theta} \quad & \frac{1}{n}(1 \quad \beta^n) \\
e_{zz} \quad & \frac{1}{n}[1 \quad (1 \quad d)^n] \\
e_{r\theta} \quad & e_{\theta z} \quad e_{zr} \quad 0
\end{aligned}
\tag{4}
$$

The stress-strain relation for thermo elastic isotropic material are given by [17]:

$$T_{ij} = \lambda\delta_{ij}I_1 + 2\mu e_{ij} - \xi\Theta\delta_{ij}, \quad (i, j = 1, 2, 3) \tag{5}$$

where T_{ij} are the stress components, λ and μ are Lame's constants, $I_1 = e_{kk}$ is the first strain invariant, δ_{ij} is the Kronecker's delta, $\xi = \alpha(3\lambda + 2\mu)$, α being the coefficient of thermal expansion, and Θ is the temperature. Further, Θ has to satisfy:

$$\nabla^2\Theta = 0$$

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}r^2} \quad \frac{1}{r}\frac{\mathrm{d}^2\Theta}{\mathrm{d}r^2} \quad \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} \quad r\frac{\mathrm{d}\Theta}{\mathrm{d}r} \quad 0$$

or

$$\frac{\mathrm{d}\Theta}{\mathrm{d}r} \quad \frac{k}{r}$$

which has solutions:

$$\Theta = k_1 (\log r + k_2) \tag{6}$$

where k_1 and k_2 are constants of integration and can determined from the boundary condition.

Equation (5) for this problem become:

$$
\begin{aligned}
T_{rr} \quad & \frac{2\lambda\mu}{\lambda \quad 2\mu}(e_{rr} \quad e_{\theta\theta}) \quad 2\mu e_{rr} \quad \frac{2\mu\xi\Theta}{\lambda \quad 2\mu} \\
T_{\theta\theta} \quad & \frac{2\lambda\mu}{\lambda \quad 2\mu}(e_{rr} \quad e_{\theta\theta}) \quad 2e_{\theta\theta} \quad \frac{2\mu\xi\Theta}{\lambda \quad 2\mu} \\
T_{r\theta} \quad & T_{\theta z} \quad T_{zr} \quad T_{zz} \quad 0
\end{aligned}
\tag{7}
$$

Substituting eqs. (4) in eq. (5), the strain components in terms of stresses are obtained as [15]:

$$
\begin{aligned}
e_{rr} \quad & \frac{\partial u}{\partial r} \quad \frac{1}{2} \quad \frac{\partial u}{\partial r}^{2} \quad \frac{1}{2}[1 \quad (r\beta \quad \beta)^2] \quad \frac{1}{E} \quad T_{rr} \quad \frac{1 \quad C}{2 \quad C} T_{\theta\theta} \quad \alpha\Theta \\
e_{\theta\theta} \quad & \frac{u}{r} \quad \frac{u^2}{2r^2} \quad \frac{1}{2}(1 \quad \beta^2) \quad \frac{1}{E} \quad T_{\theta\theta} \quad \frac{1 \quad C}{2 \quad C} T_{rr} \quad \alpha\Theta \\
e_{zz} \quad & \frac{\partial w}{\partial z} \quad \frac{1}{2} \quad \frac{\partial w}{\partial z}^{2} \quad \frac{1}{2}[1 \quad (1 \quad \mathrm{d})^2] \quad \frac{1 \quad C}{E(2 \quad C)}(T_{rr} \quad T_{\theta\theta}) \quad \alpha\Theta \\
& e_{r\theta} \quad e_{\theta z} \quad e_{zr} \quad 0
\end{aligned} \tag{8}
$$

where E is the Young's modulus and C is the compressibility factor of the material in term of Lame's constant, and are given by $E \quad \mu(3\lambda \quad 2\mu)/(\lambda \quad \mu)$ and $C \quad 2\mu/(\lambda \quad 2\mu)$. Substituting eqs. (4) in eqs. (7), we get:

$$
\begin{aligned}
T_{rr} \quad & \frac{2\mu}{n} \quad 3 \quad 2C \quad \beta^n \; 1 \quad C \quad (2 \quad C) \; \frac{r\beta}{\beta} \quad 1^{\;\;n} \quad \frac{nC\xi\Theta}{2\mu\beta^n} \\
T_{\theta\theta} \quad & \frac{2\mu}{n} \quad 3 \quad 2C \quad \beta^n \; 2 \quad C \quad (1 \quad C) \; \frac{r\beta}{\beta} \quad 1^{\;\;n} \quad \frac{nC\xi\Theta}{2\mu\beta^n}
\end{aligned} \tag{9}
$$

and

$$T_{r\theta} \quad T_{\theta z} \quad T_{zr} \quad T_{zz} \quad 0$$

All equations of equilibrium are satisfied except:

$$\frac{\mathrm{d}}{\mathrm{d}r}(rT_{rr}) \quad T_{\theta\theta} \quad \rho\omega^2 r^2 \quad 0 \tag{10}$$

where ρ is the density of the material of the rotating disc.

The temperature satisfying Laplace eq. (6) with boundary condition:

$$\Theta \quad \Theta_0 \;\text{ at }\; r \quad a,$$
$$\Theta \quad 0 \;\text{ at }\; r \quad b$$

where Θ_0 is constant, given by [17]:

$$k_1 \quad \frac{\Theta}{\log\frac{a}{b}} \;\text{ and }\; k_2 \quad \log b$$

Substituting k_1 and k_2 form eq. (6), we get:

$$\Theta_1 \quad \frac{\Theta_0 \log\frac{r}{b}}{\log\frac{a}{b}} \tag{11}$$

Using eqs. (9) and (11) in eq. (10), we get a non-linear differential equation in β as:

$$(2 \quad C)n\beta^{n \quad 1}P(P \quad 1)^{n \quad 1}\frac{\mathrm{d}P}{\mathrm{d}\beta} \quad \frac{n\rho\omega^2 r^2}{2\mu} \quad \frac{nC\xi\overline{\Theta}_0}{2\mu}$$
$$\beta^n\{1 \quad (P \quad 1)^n \quad np[1 \quad C \quad (2 \quad C)(P \quad 1)^n]\} \tag{12}$$

where $\overline{\Theta}_0 \quad \Theta_0/\log(a/b)$ and $r\beta' = \beta P$ (P is function of β and β is function of r).

From eq. (12), the turning points of β are $P = -1$ and $\quad \infty$.

The boundary conditions are:

$$u = 0 \text{ at } r = a \text{ and } T_{\mathrm{rr}} = 0 \text{ at } r = b \tag{13}$$

Solution through the principal stresses

For finding the plastic stress, the transition function is taken through the principal stress (see Seth [7, 8], Hulsurkar [9], and Gupta *et al*. [10-14, 16]) at the transition point $P \quad \infty$. We take the transition function R as:

$$R \quad \frac{n}{2\mu}(T_{\theta\theta} \quad C\xi\Theta) \quad (3 \quad 2C) \quad \beta^n[2 \quad C \quad (1 \quad C)(P \quad 1)^n] \quad \frac{nC\xi\Theta}{\mu} \tag{14}$$

Taking the logarithmic differentiation of eq. (14) with respect to r and using eq. (12), we get:

$$\frac{\mathrm{d}(\log R)}{\mathrm{d}r} \quad \frac{\beta^n \dfrac{1 \quad C}{2 \quad C}\left\{1 \quad (P \quad 1)^n \quad n(1 \quad C)P \quad \dfrac{n\rho\omega^2 r^2}{2\mu\beta^n} \quad \dfrac{nC\xi\overline{\Theta}_0(3 \quad 2C)}{\mu(4 \quad 2C)\beta^n}\right\} \quad (2 \quad C)nP\beta^n}{r\left\{3 \quad 2C \quad \beta^n[2 \quad C \quad (1 \quad C)(P \quad 1)^n] \quad \dfrac{nC\xi\Theta}{2\mu}\right\}} \tag{15}$$

Taking the asymptotic value of eq. (15) at $P \quad \infty$ and integrating, we get:

$$R \quad K_1 r^{\frac{1}{2 \quad C}} \tag{16}$$

where K_1 is a constant of integration, which can be determine by boundary condition.

From eqs. (14) and (16), we have:

$$T_{\theta\theta} \quad \frac{2\mu}{n} A_1 r^{\frac{1}{2 \quad C}} \quad \frac{C\xi\Theta_0 \log\frac{r}{b}}{\log\frac{a}{b}} \tag{17}$$

Substituting eq. (17) in eq. (10) and integrating, we get:

$$T_{\theta\theta} \quad \frac{2\mu(2 \quad C)}{n(1 \quad C)} A_1 r^{\frac{1}{2 \quad C}} \quad \frac{C\xi\Theta_0 \log\frac{r}{b}}{\log\frac{a}{b}} \quad \frac{C\xi\Theta_0}{\log\frac{a}{b}} \quad \frac{\rho\omega^2 r^2}{3} \quad \frac{K_2}{r} \tag{18}$$

where K_2 is a constant of integration, which can be determine by boundary condition.

Substituting eq. (17) and (18) in second equation of eqs. (8), we get:

$$\beta \quad \sqrt{1 \quad \frac{2(1 \quad C)}{E(2 \quad C)} \quad \frac{\rho\omega^2 r^2}{3} \quad \frac{\alpha E\Theta_0(2 \quad C)}{\log\frac{a}{b}} \quad \frac{2(2 \quad C)\alpha E\Theta_0 \log\frac{r}{b}}{\log\frac{a}{b}} \quad \frac{B_1}{r}} \tag{19}$$

where $C\xi = \alpha E(2 - C)$.

Substituting eq. (19) in eq. (2), we get:

$$u \quad r \quad r\sqrt{1 \quad \frac{2(1 \quad C)}{E(2 \quad C)} \quad \frac{\rho\omega^2 r^2}{3} \quad \frac{\alpha E\Theta_0(2 \quad C)}{\log\frac{a}{b}} \quad \frac{2(2 \quad C)\alpha E\Theta_0 \log\frac{r}{b}}{\log\frac{a}{b}} \quad \frac{B_1}{r}} \tag{20}$$

where $E = 2\mu(3 - 2C)/(2 - C)$ is the Young's modulus in term of compressibility factor.

Using boundary condition (13) in eqs. (18) and (20), we get:

$$K_1 \quad \frac{\rho\omega^2 n(1 \quad C)(b^3 \quad a^3)}{6\mu(2 \quad C) b^{\frac{1 \quad C}{2 \quad C}}} \quad \frac{\alpha E\Theta_0 n(1 \quad C)(b \quad a)}{2\mu \log\frac{a}{b} b^{\frac{1 \quad C}{2 \quad C}}} \quad \frac{\alpha E\Theta_0 na}{\mu b^{\frac{1 \quad C}{2 \quad C}}} \tag{21}$$

$$K_2 \quad \frac{\rho\omega^2 a^3}{3} \quad \frac{\alpha E\Theta_0(2 \quad C)a}{\log\frac{a}{b}} \quad \frac{2(2 \quad C)\alpha E\Theta_0 a}{1 \quad C} \tag{22}$$

Substituting eqs. (21) and (22) in eqs. (17), (18), and (20), respectively, we get the transitional stresses and displacement as:

$$T_{\theta\theta} \quad \frac{\rho\omega^2(1 \quad C)(b^3 \quad a^3)}{3r(2 \quad C)} \quad \frac{r}{b}^{\frac{1 \quad C}{2 \quad C}}$$

$$\alpha E\Theta_0(2 \quad C)\ \frac{\log\frac{r}{b}}{\log\frac{a}{b}} \quad \frac{2a}{(2 \quad C)r} \quad \frac{r}{b}^{\frac{1 \quad C}{2 \quad C}} \quad \frac{(1 \quad C)(b \quad a)}{r(2 \quad C)\log\frac{a}{b}} \quad \frac{r}{b}^{\frac{1 \quad C}{2 \quad C}} \tag{23}$$

$$T_{\mathrm{rr}} \quad \frac{\rho\omega^2}{3r} \quad (b^3 \quad a^3) \quad \frac{r}{b}^{\frac{1 \quad C}{2 \quad C}} \quad r^3 \quad a^3$$

$$\frac{\alpha E\Theta_0(2 \quad C)}{\log\frac{a}{b}} \quad \log\frac{r}{b} \quad \frac{b \quad a}{r} \quad \frac{r}{b}^{\frac{1 \quad C}{2 \quad C}} \quad \frac{a}{r} \quad 1$$

$$\frac{2\alpha E\Theta_0(2 \quad C)}{1 \quad C} \quad \frac{a}{r} \quad \frac{a}{r} \quad \frac{r}{b}^{\frac{1 \quad C}{2 \quad C}} \tag{24}$$

and

$$u \quad r \quad r\sqrt{1 \quad \frac{2}{E}\frac{1 \quad C}{2 \quad C} \quad \begin{matrix} \dfrac{\rho\omega^2(r^3 \quad a^3)}{3r} \quad \dfrac{\alpha E\Theta_0(2 \quad C)(r \quad a)}{r\log\frac{a}{b}} \\ \dfrac{2(2 \quad C)\alpha E\Theta_0}{1 \quad C} \quad \dfrac{\log\frac{r}{b}}{\log\frac{a}{b}} \quad \dfrac{a}{r} \end{matrix}} \tag{25}$$

From eqs. (23) and (24), we get:

$$\left|T_{\mathrm{rr}} \quad T_{\theta\theta}\right| \quad \left| \frac{\rho\omega^2}{3} \quad \frac{b^3 \quad a^3}{r(2 \quad C)} \quad \frac{r}{b}^{\frac{1 \quad C}{2 \quad C}} \quad r^2 \quad \frac{a^3}{r} \right.$$

$$\left. \alpha E\Theta_0 \ 2\frac{a}{r}\frac{2 \quad C}{1 \quad C} \quad \frac{2a}{(1 \quad C)r} \quad \frac{r}{b}^{\frac{1 \quad C}{2 \quad C}} \quad \frac{b \quad a}{r} \quad \frac{r}{b}^{\frac{1 \quad C}{2 \quad C}} \quad \frac{2 \quad C}{\log\frac{a}{b}}\frac{a \quad r}{r} \right| \tag{26}$$

From eq. (26), it is seen that $|T_{\rm rr} \quad T_{\theta\theta}|$ is maximum at the internal surface (that is at $r = a$), therefore yielding of the disc takes place at the internal surface of the disc and eq. (26) can be written as:

$$|T_{\rm rr} \quad T_{\theta\theta}|_{r \; a} \quad \left| \frac{\rho\omega^2(b^3 \quad a^3)}{3(2 \quad C)a} \left(\frac{a}{b}\right)^{\frac{1 \; C}{2 \; C}} \right.$$

$$\left. \alpha E\Theta_0 \frac{b \quad a}{a\log\frac{a}{b}} \left(\frac{a}{b}\right)^{\frac{1 \; C}{2 \; C}} \quad \frac{2}{1 \quad C}\left(\frac{a}{b}\right)^{\frac{1 \; C}{2 \; C}} \quad \frac{2(2 \quad C)}{1 \quad C} \right| \quad Y$$

where Y is the yielding stress.

The angular speed $\Omega_{\rm i}$ necessary for initial yielding is given by:

$$\Omega_{\rm i}^2 \quad \frac{\rho\omega_{\rm i}^2 b^2}{Y} \quad \left| \frac{3(2 \quad C)\left(\frac{a}{b}\right)^{\frac{1}{2 \; C}}}{1 \quad \frac{a^3}{b^3}} \right|$$

$$\left| \frac{3(2 \quad C)}{1 \quad \frac{a^3}{b^3}} \frac{\alpha E\Theta_0}{Y} \frac{1 \quad \frac{a}{b}}{\log\frac{a}{b}} \quad \frac{2(2 \quad C)}{1 \quad C}\left(\frac{a}{b}\right)^{\frac{1}{2 \; C}} \quad \frac{2}{1 \quad C}\frac{a}{b} \right| \tag{27}$$

and

$$\omega_{\rm i} \quad \frac{\Omega_{\rm i}}{b}\sqrt{\frac{Y}{\rho}}$$

The disc becomes fully plastic ($C \quad 0$) at the external surface and eq. (26) becomes:

$$|T_{\rm rr} \quad T_{\theta\theta}|_{r \; b} \quad \left| \frac{\rho\omega^2(b^3 \quad a^3)}{6b} \quad \alpha E\Theta_0 \frac{b \quad a}{b\log\frac{a}{b}} \quad 2\frac{a}{b} \right| \quad Y$$

where $E = 3\mu$.

The angular $\Omega_{\rm f}$ speed required for fully plastic state is given by:

$$\Omega_{\rm f}^2 \quad \frac{\rho\omega_{\rm f}^2 b^2}{Y} \quad \left| \frac{6}{1 \quad \frac{a^3}{b^3}} \right| \left| \frac{6}{1 \quad \frac{a^3}{b^3}} \frac{\alpha E\Theta_0}{Y} \frac{1 \quad \frac{a}{b}}{\log\frac{a}{b}} \quad 2\frac{a}{b} \right| \tag{28}$$

where

$$\omega_{\mathrm{f}} \quad \frac{\Omega_{\mathrm{f}}}{b}\sqrt{\frac{Y}{\rho}}$$

We introduce the following non-dimensional components:

$$R \quad \frac{r}{b}, R_0 \quad \frac{a}{b}, \sigma_{\mathrm{r}} \quad \frac{T_{\mathrm{rr}}}{Y}, \sigma_{\theta} \quad \frac{T_{\theta\theta}}{Y}, \bar{u} \quad \frac{u}{b}, \Theta_1 \quad \frac{\alpha E \Theta_0}{Y}, \Omega_{\mathrm{i}}^2 \quad \frac{\rho\omega_{\mathrm{i}}^2 b^2}{Y}, \text{and } \Omega_{\mathrm{f}}^2 \quad \frac{\rho\omega_{\mathrm{f}}^2 b^2}{Y}$$

Elastic-plastic transitional stresses, angular speed and displacement from eq. (23)-(25), and (27) in non-dimensional form become:

$$\sigma_{\theta} \quad \frac{\Omega_{\mathrm{i}}^2}{3R} \; \frac{(1 \quad R_0^3)(1 \quad C)}{2 \quad C} R^{\frac{1 \quad C}{2 \quad C}}$$

$$\Theta_1(2 \quad C) \; \frac{(1 \quad R_0)(1 \quad C)}{(2 \quad C)R \log R_0} R^{\frac{1 \quad C}{2 \quad C}} \quad \frac{\log R}{\log R_0} \quad \frac{2R_0}{(2 \quad C)R} R^{\frac{1 \quad C}{2 \quad C}} \tag{29}$$

$$\sigma_{\mathrm{r}} \quad \frac{\Omega_{\mathrm{i}}^2}{3R} \; (1 \quad R_0^3) R^{\frac{1 \quad C}{2 \quad C}} \quad R^3 \quad R_0^3$$

$$\frac{\Theta_1(2 \quad C)}{\log R_0} \; \log R \quad \frac{1 \quad R_0}{R} R^{\frac{1 \quad C}{2 \quad C}} \quad \frac{R_0}{R} \quad 1 \quad \frac{2\Theta_1(2 \quad C)}{1 \quad C} \; \frac{R_0}{R} \quad \frac{R_0}{R} R^{\frac{1 \quad C}{2 \quad C}} \tag{30}$$

$$\Omega_{\mathrm{i}}^2 \quad \left| \frac{3(2 \quad C)}{1 \quad R_0^3} R_0^{\frac{1}{2 \quad C}} \right| \quad \left| \frac{3\Theta_1(2 \quad C)}{1 \quad R_0^3} \; \frac{1 \quad R_0}{\log R_0} \quad \frac{2(2 \quad C)}{1 \quad C} R_0^{\frac{1}{2 \quad C}} \quad \frac{2}{1 \quad C} R_0 \right| \tag{31}$$

and

$$\bar{u} \quad R \quad R\sqrt{1 \quad 2\frac{Y}{E}\frac{1 \quad C}{2 \quad C} \quad \begin{matrix} \frac{\Omega_{\mathrm{i}}^2}{3R}(R^3 \quad R_0^3) \\ \frac{\Theta_1(2 \quad C)(R \quad R_0)}{R \log R_0} \quad \frac{2(2 \quad C)\Theta_1}{1 \quad C} \; \frac{\log R}{\log R_0} \quad \frac{R_0}{R} \end{matrix}} \tag{32}$$

Stresses, displacement, and angular speed for fully plastic state (C , are obtained from eqs. (29), (30), (32), and (28) as:

$$\sigma_\theta \quad \frac{\Omega_\mathrm{f}^2}{3R} \quad \frac{1 \quad R_0^3}{2}\sqrt{R} \quad 2\Theta_1 \quad \frac{1 \quad R_0}{2R\log R_0}\sqrt{R} \quad \frac{\log R}{\log R_0} \quad \frac{R_0}{R}\sqrt{R} \tag{33}$$

$$\sigma_\mathrm{r} \quad \frac{\Omega_\mathrm{f}^2}{3R}[(1 \quad R_0^3)\sqrt{R} \quad R^3 \quad R_0^3]$$

$$\frac{2\Theta_1}{\log R_0} \quad \log R \quad \frac{1 \quad R_0}{r}\sqrt{R} \quad \frac{R_0}{R} \quad 1 \quad 4\Theta_1 \quad \frac{R_0}{R} \quad \frac{R_0}{R}\sqrt{R} \tag{34}$$

$$\bar{u} \quad R \quad R\sqrt{1 \quad \frac{Y}{E} \quad \frac{\Omega_\mathrm{f}^2}{3R}(R^3 \quad R_0^3) \quad \frac{2\Theta_1(R \quad R_0)}{R\log R_0} \quad 4\Theta_1 \quad \frac{\log R}{\log R_0} \quad \frac{R}{R_0}} \tag{35}$$

and

$$\Omega_\mathrm{f}^2 \quad \left|\frac{6}{1 \quad R_0^3}\right| \quad \left|\frac{6\Theta_1}{1 \quad R_0^3} \quad \frac{1 \quad R_0}{\log R_0} \quad 2R_0\right| \tag{36}$$

Particular case

When there is no thermal effect ($\Theta_1 = 0$), the transitional stresses from eq. (29) to (32) become:

$$\sigma_\theta \quad \frac{\Omega_\mathrm{i}^2}{3R} \quad \frac{(1 \quad R_0^3)(1 \quad C)}{2 \quad C}R^{\frac{1 \quad C}{2 \quad C}} \tag{37}$$

$$\sigma_\mathrm{r} \quad \frac{\Omega_\mathrm{i}^2}{3R} \quad (1 \quad R_0^3)R^{\frac{1 \quad C}{2 \quad C}} \quad R^3 \quad R_0^3 \tag{38}$$

$$\bar{u} \quad R \quad R\sqrt{1 \quad 2\frac{Y}{E}\frac{1 \quad C}{2 \quad C} \quad \frac{\Omega_\mathrm{i}^2}{3R}(R^3 \quad R_0^3)} \tag{39}$$

where

$$\Omega_\mathrm{i}^2 \quad \frac{3(2 \quad C)}{1 \quad R_0^3}R_0^{\frac{1}{2 \quad C}} \tag{40}$$

For fully plastic state stresses, displacement, and angular speed from eq. (33) to (36) becomes:

$$\sigma_\theta \quad \frac{\Omega_\mathrm{f}^2}{3R}\,\frac{1 \quad R_0^3}{2}\sqrt{R} \tag{41}$$

$$\sigma_\mathrm{r} \quad \frac{\Omega_\mathrm{f}^2}{3R}[(1 \quad R_0^3)\sqrt{R} \quad R^3 \quad R_0^3] \tag{42}$$

$$\bar{u} \quad R \quad R\sqrt{1 \quad \frac{Y}{E}\,\frac{\Omega_\mathrm{f}^2}{3R}(R^3 \quad R_0^3)} \tag{43}$$

where

$$\Omega_\mathrm{f}^2 \quad \frac{\rho\omega_\mathrm{f}^2 b^2}{Y} \quad \left|\frac{6}{1 \quad R_0^3}\right| \tag{44}$$

These equation are same as obtained by Gupta and Pankaj [15].

Results and discussion

For calculating the stresses, angular speed, and displacement based on the above analysis, the following values have been taken: C = 0.00, 0.25, 0.5, and 0.75, E/Y = 1/2 and 2, Θ_0 = 0 and 700 °F, $\alpha = 5.0 \quad 10^{-5}$ deg F^{-1} (for methyl methacrylate) [18], $\Theta_1 = \alpha E\Theta_0/Y$ = 0, 0.0175, and 0.07 for E/Y = 1/2 and 2, and θ_0 = 0 and 700 °F, respectively.

Curves have been drawn in fig. 2 between angular speed Ω_i^2 required for initial yielding and various radii ratios $R_0 = a/b$ for C = 0, 0.25, 0.5, and 0.75 at Θ_1 = 0, 0.0175, and 0.07. It has been observed that in the absence of thermal effect the rotating disc made of incompressible material with inclusion require higher angular speed to yield at the internal surface as compare to disc made of compressible material and a much higher angular speed is required to yield with the increase in radii ratio. With the introduction of thermal effects, lesser angular speed is required to yield at the internal surface. It can also be seen from tab. 1 that for compressible material higher percentage increased in angular speed is required to become fully plastic as compared to rotating disc made of incompressible material.

In figs. 3 and 4, curves have been drawn for stresses and displacement with respect to radii ratio $R = r/b$ for elastic-plastic transition and fully plastic state, respectively. It has been seen that temperature has a quite effect on radial and circumferential stresses *i. e.* with the introduction of thermal effect it decrease the value of radial and circumferential stress at the internal surface for transitional state, whereas from fig. 4, it can been seen that thermal effect increases the values of radial and circumferential stress at the internal surface for fully-plastic state.

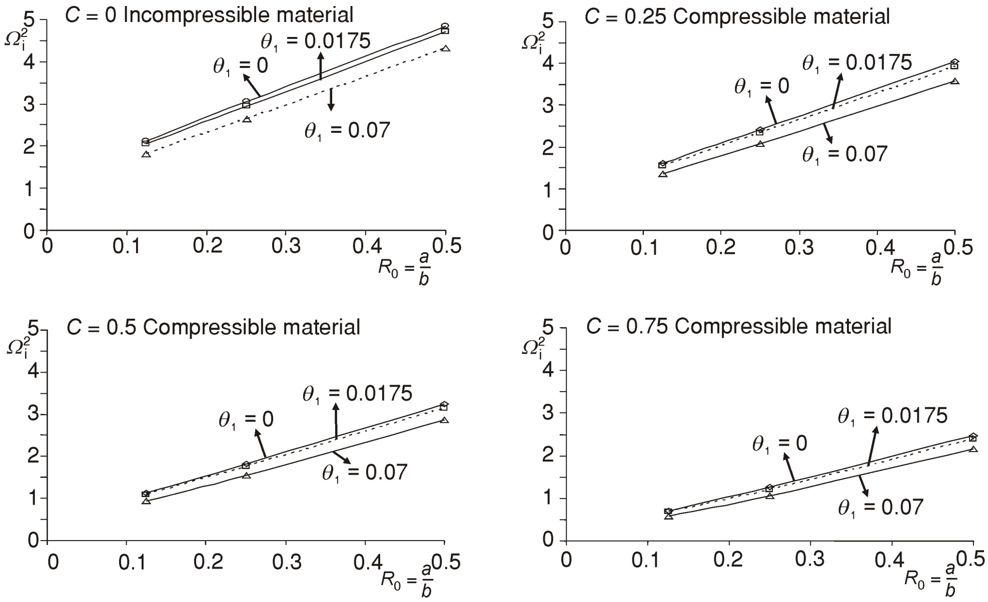

Figure 2. Angular speed required for initial yielding at the internal surface of the rotating disc

Table 1. Angular speed required for initial yielding and fully plastic state

	Different values of temperatures, Θ	Compressibility of material, C	Angular speed required for initial yielding, Ω_i^2	Angular speed required for fully-plastic state Ω_f^2	Percentage increase in angular speed $\sqrt{\frac{\Omega_f^2}{\Omega_i^2}}$ 1 100 [%]
0.5 R 1.0	0	0	4.848732	6.857143	18.92071152
	0.0175	0	4.719678	7.063705	22.33762967
	0.07	0	4.317334	7.682293	33.40392435
	0	0.25	4.037701	6.857143	30.31804128
	0.0175	0.25	3.926953	7.063705	34.11840831
	0.07	0.25	3.581684	7.682293	46.46463551
	0	0.5	3.239797	6.857143	45.4831515
	0.0175	0.5	3.147226	7.063705	49.81396203
	0.07	0.5	2.858767	7.682293	63.94078954
	0	0.75	2.461496	6.857143	66.90601558
	0.0175	0.75	2.387026	7.063705	72.02346519
	0.07	0.75	2.154855	7.682293	88.82842534

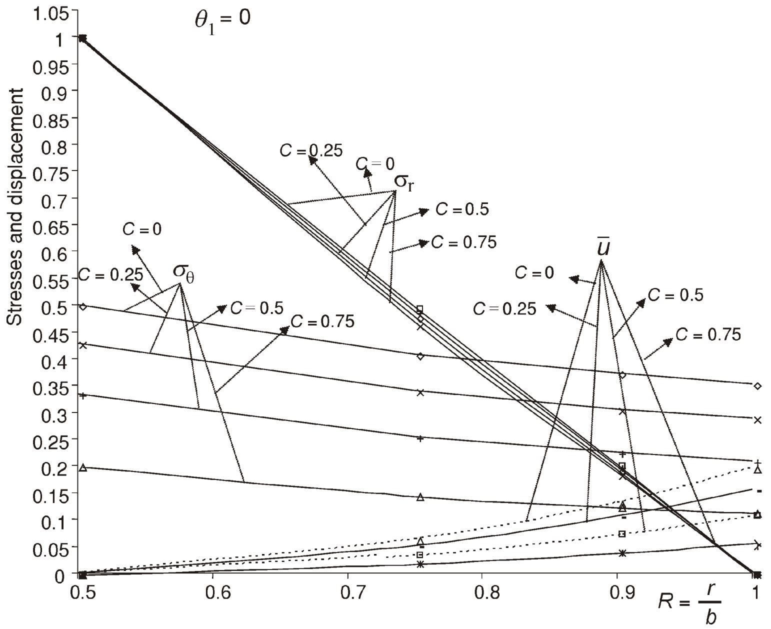

Figure 3a. Stresses at the elastic plastic transition state

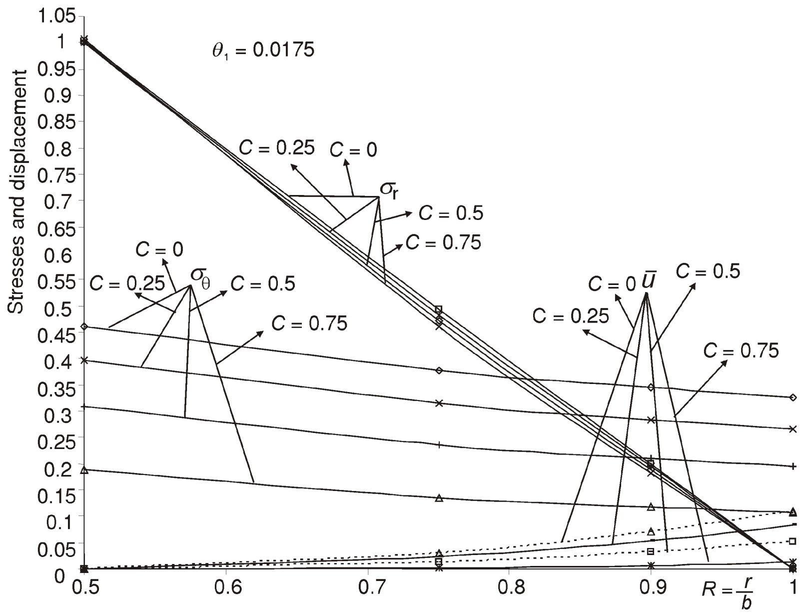

Figure 3b. Stresses at the elastic plastic transition state

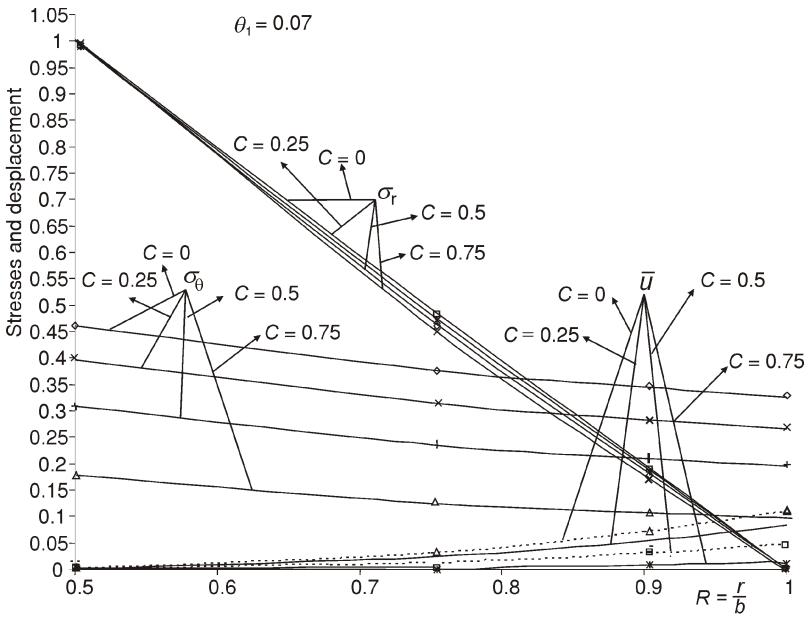

Figure 3c. Stresses at the elastic plastic transition state

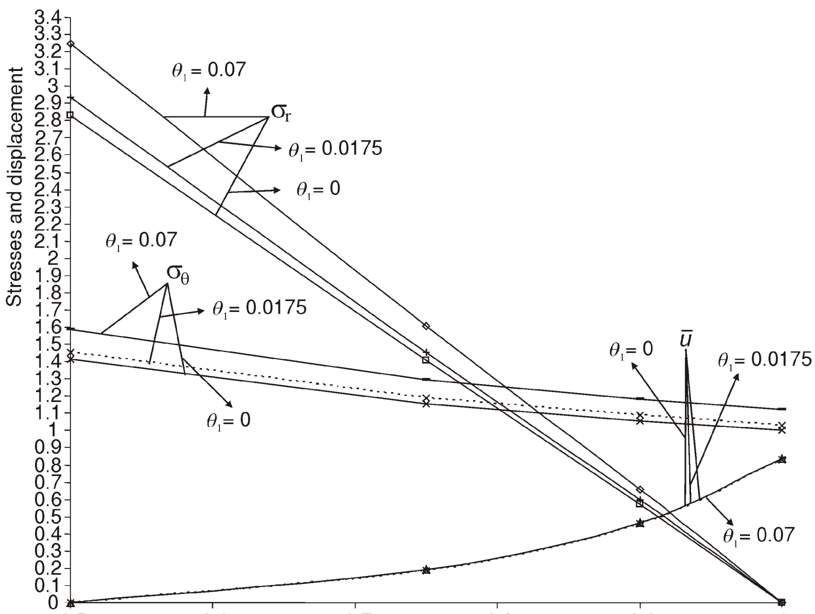

Figure 4. Stresses at the fully plastic state

Acknowledgment

The authors are grateful to the referee for his critical comments, which led to a significant improvement of the paper.

Nomenclature

a, b	– internal and external radii of the disc, [m]
C	– compressibility factor, [–]
K_1, K_2, k_1, k_2	– constants of integration, [–]
T_{ij}, e_{ij}	– stress [$kgm^{-1}s^{-2}$] and strain rate tensor
u, v, w	– displacement components, [m]
Y	– yield stress, [$kgm^{-1}s^{-2}$]

Greek letters

Θ	– temperature, [°F]
σ_r	– radial stress component (T_{rr}/Y), [–]
σ_θ	– circumferential stress component ($T_{\theta\theta}/Y$), $\Theta_1 = \alpha E \Theta_0 / Y$, [–]
ρ	– density of material, [kgm^{-3}]
Ω^2	– $\rho\omega^2 b^2/E$ (speed factor), $R = r/b$, $R_0 = a/b$ (radii ratio), [–]
ω	– angular speed of rotation, [s^{-1}]

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Paper submitted: April 14, 2006
Paper revised: December 12, 2006
Paper accepted: December 20, 2006