ON GENERALIZED HYDROMAGNETIC THERMOSOLUTAL CONVECTION: THE DUFOUR-EFFECT

by

Hari MOHAN, Pardeep KUMAR , and Pawittar SINGH

Original scientific paper
UDC: 536.25:537.87:532.54
BIBLID: 0354-9836, *9* (2005), 1, 139-150

The effect of uniform magnetic field on the Dufour-driven thermosolutal convection of an electrically conducting fluid completely confined in an arbitrary region bounded by rigid walls is considered. Some general qualitative results concerning the character of marginal state, stability of oscillatory motions and limitations on the oscillatory motions of growing amplitude, are derived. The results for the horizontal layer geometry in the present case follow as a consequence.

Key words: *Dufour-driven thermosolutal convection, Rayleigh numbers, Lewis numbers, Prandtl numbers, Chandrasekhar number*

Introduction

The stability properties of binary fluids are quite different from pure fluids because of Soret and Dufour [1, 2] effects. An externally imposed temperature gradient produces a chemical potential gradient and the phenomenon, known as the Soret effect, arises when the mass flux contains a term that depends upon the temperature gradient. The analogous effect that arises from a concentration gradient dependent term in the heat flux is called the Dufour effect. Although it is clear that the thermosolutal and Soret-Dufour problems are quite closely related, their relationship has never been carefully elucidated. They are in fact, formally identical and this is done by means of a linear transformation that takes the equations and boundary conditions for the latter problem into those for the former. Recently, Hari Mohan [3] mollified the nastily behaving governing equations of Dufour-driven thermosolutal convection of the Veronis [4] type by the construction of an appropriate linear transformation and derived the desired results concerning the linear growth rate and the behavior of oscillatory motion on the lines suggested by Banerjee *et. al* [5].

Almost all the papers that are written on the subject are confined to horizontal layer geometry on account of complexity of the problem for arbitrary geometry. However, there do exist a class of results in the domain of hydrodynamic and hydromagnetic stability theory that possess the sparks of their generalization to containers of arbitrary shape [6].

The present paper investigates the instability of Dufour-driven thermosolutal convection of an electrically conducting fluid completely confined in an arbitrary region bounded by rigid walls in the presence of a uniform magnetic field applied in an arbitrary direction and derives some general qualitative results concerning the character of marginal state, stability of oscillatory motions and limitations on the oscillatory motions of growing amplitude. The results for the horizontal layer geometry in the present case follow as a consequence.

Mathematical formulation and analysis

The relevant governing non-dimensional linearized perturbation equations in the present case with time dependence of the form exp(pt) ($p = p_r + ip_i$) are given by:

$$\frac{p}{\sigma}\vec{q} \quad (p) \quad \text{curl curl}\,\vec{q} \quad R_T\theta\hat{\beta} \quad R_S\phi\hat{\beta} \quad Q(\text{curl}\,\vec{h}) \quad \hat{\ell} \tag{1}$$

$$(\quad ^2 \quad p)\theta \quad R_3\gamma \quad ^2\phi \quad \vec{q} \quad \hat{\beta} \tag{2}$$

$$(\tau \quad ^2 \quad p)\phi \quad \vec{q} \quad \hat{\beta} \tag{3}$$

$$\text{curl curl}\,\vec{h} \quad \frac{p\sigma_1\vec{h}}{\sigma} \quad \text{curl}(\vec{q} \quad \hat{\ell}) \tag{4}$$

and

$$\vec{q} \quad 0 \quad \vec{h} \tag{5}$$

In the above equations $\vec{q}(x,y,z), \rho(x,y,z), \theta(x,y,z), \phi(x,y,z)$, and $\vec{h}(x,y,z)$ respectively denote the perturbed velocity, pressure, temperature, concentration, and magnetic field and are complex valued functions defined on V, $R_T \quad g\alpha\beta d^4/\kappa\nu$ is the thermal Reyleigh number, $R_S \quad g\alpha\,\beta\,d^4/\kappa\,\nu$ is the concentration Rayleigh number, $Q \quad \mu e H_0^2 d^2/4\pi\rho_0\nu\eta$ is the Chandrasekhar number, $\tau \quad \kappa \,/\, \kappa$ is the Lewis number, $V > 0$ is referred here as Dufour number, and $\hat{\beta}$ is a unit vertical vector. Further, with d as the characteristic length, the equations have been cast into dimensionless forms by using the scale factors κ/d, d^2/κ, βd, $\rho\upsilon\kappa/d^2$, $\beta' d$, and $\kappa H_0/\eta$ for velocity, time, temperature, pressure, concentration and magnetic field respectively.

Associated with the system of eqs. (1)-(5) is a set of homogeneous and time independent boundary conditions. We shall limit our consideration to the region γ completely confined by rigid walls, which may be thermal, and concentration-wise conducting or insulating and to see the case when the electrical conductivity of the wall is large in comparison to the field (see [6]). Thus we seek solutions of eqs. (1)-(5) in the simply connected subset V of R_3 subject to the following boundary conditions:

$$\text{either} \quad \vec{q} \quad 0 \quad \theta \quad \varphi \quad \hat{n} \quad \text{curl}\,\vec{h} \quad \text{on} \quad S \tag{6}$$

$$\text{or} \quad \vec{q} \quad 0 \quad \theta\vec{n} \quad \phi\vec{n} \quad \vec{n} \quad \text{curl}\,\vec{h} \quad \text{on} \quad S \tag{7}$$

where $\hat{n}$ is a unit vector in the direction of the normal to boundary surface S.

Equations (1)-(5) together with boundary conditions (6)-(7) constitute an eigenvalue problem for p for given values of other parameters. The system is stable, neutral or unstable according to the sign of p_r (negative, zero or positive). Further:

(a) p_i 0 and p_r 0 describe oscillatory motion of neutral or growing amplitude,

(b) $R_T < 0$, $R_S < 0$, $\gamma > 0$, and $Q = 0$ describe Stern [7] thermohaline configuration in the present generalized set up which for convenience is epitomized in abbreviated form as GSTHC, and

(c) $\Gamma \quad |R_S|/|R_T|$ takes care of initial density gradient of the configuration.

Finally if $p_r = 0 \quad p_i = 0$, then the principle of exchange of stabilities (PES) is valid, otherwise, we have overstability.

We now, prove the following lemmas and theorems.

Lemma 1: (Poincare : Inequality) – If $f(x, y, z)$ is any smooth function which vanishes on S, and l is the smallest distance between two parallel planes which just contains V, then there exists a constant (> 2) such that:

$$\int_V | \quad F|^2 \mathrm{dV} \quad \frac{\lambda}{\ell^2} \quad |f|^2 \mathrm{dV} \tag{8}$$

Proof: See Joseph [8].

Lemma 2: If $(p, \vec{q}, \vec{h}, \theta, \phi)$ is a non-trivial solution of eq. (1)-(5) together with either of the boundary conditions, then the following integral relations hold:

$$\int_V \vec{q}^* \quad \text{curl}\,\text{curl}\,\vec{q}\,\mathrm{dV} \quad \int_V |\text{curl}\,\vec{q}|^2 \mathrm{dV} \tag{9}$$

$$\int_V \vec{q}^* \quad \text{curl}\,\text{curl}(\mathrm{q} \quad \hat{\ell})\mathrm{dV} \quad \int_V \text{curl}(\vec{q} \quad \hat{\ell})\,\text{curl}\,\vec{q}^*\,\mathrm{dV} \tag{10}$$

$$\int_V \vec{q}^* \quad \text{curl}\,\text{curl}(\theta\hat{\beta})\,\mathrm{dV} \quad 0 \quad \int_V \vec{q}^* \quad \text{curl}\,\text{curl}(\phi\hat{\beta})\,\mathrm{dV} \tag{11}$$

$$\int_V \vec{q}^* \quad [(\text{curl}\,\vec{h}) \quad \hat{\ell}]\mathrm{dV} \quad \int_V \vec{h} \quad \text{curl}\,\text{curl}(\vec{q}^* \quad \hat{\ell})\,\mathrm{dV} \tag{12}$$

$$\int_V \vec{q}^* \quad [\hat{\ell}\,\text{curl}\,\text{curl}\,\text{curl}\,\vec{h}]\mathrm{dV} \quad \int_V \text{curl}\,\text{curl}\,\vec{h} \quad \text{curl}(\vec{q}^* \quad \hat{\ell})\,\mathrm{dV} \tag{13}$$

$$\int_V \vec{h}^* \quad \text{curl}\,\text{curl}\,\text{curl}\,\vec{h}\,\mathrm{dV} \quad \int_V |\text{curl}\,\vec{h}|^2 \mathrm{dV} \quad \int_V \vec{h}^* \quad \text{curl}\,\text{curl}\,\vec{h}^*\,\mathrm{dV} \tag{14}$$

$$\int_V \vec{q}^* \quad (P)\mathrm{dV} \quad 0 \tag{15}$$

$$\underset{V}{}\vec{q}^*[\quad(\mathrm{div}\,\theta\hat{\beta})]\mathrm{dV}\quad 0\quad \underset{V}{}\vec{q}^*[\quad(\phi\hat{\beta})]\mathrm{dV} \tag{16}$$

$$\underset{V}{}\vec{q}^*[\quad(\hat{\ell}\ \mathrm{curl}\,\mathrm{curl}\,\vec{h})]\mathrm{dV}\quad 0 \tag{17}$$

$$\underset{V}{}\theta^*\quad {}^2\theta\mathrm{dV}\qquad \underset{V}{}|\quad\theta|^2\mathrm{dV}\quad \theta^*\quad \theta^*\mathrm{dV} \tag{18}$$

and

$$\underset{V}{}\phi^*\quad {}^2\phi\mathrm{dV}\qquad |\quad\phi|^2\mathrm{dV}\quad \phi^*\quad \phi^*\mathrm{dV} \tag{19}$$

where '*' denotes complex conjugate and $|\vec{A}|^2\quad \vec{A}\quad \vec{A}^*$ for any vector $\vec{A}$.

Proof: If $\vec{A}$, $\vec{B}$, and $\vec{C}$ are smooth vector-valued functions and Ψ is a smooth scalar-valued function on V such that $\vec{A}\quad \vec{B}$, and $\Psi\vec{C}$ vanish on S, then using Gauss' divergence theorem and the vector identities:

$$\mathrm{div}(\vec{A}\quad \vec{B})\quad \vec{B}\ \mathrm{curl}\,\vec{A}\quad \vec{A}\ \mathrm{curl}\,\vec{B}$$

and

$$\mathrm{div}(\Psi\vec{C})\qquad \Psi\vec{C}\quad \Psi\mathrm{div}\,\vec{C}$$

it follows that:

$$\underset{V}{}\vec{B}\ \mathrm{curl}\,\vec{A}\mathrm{dV}\qquad \underset{V}{}\vec{A}\ \mathrm{curl}\,\vec{B}\mathrm{dV} \tag{20}$$

and

$$\underset{V}{}\Psi\quad \vec{C}\mathrm{dV}\qquad \underset{V}{}\Psi\mathrm{div}\,\vec{C}\,\mathrm{dV} \tag{21}$$

Now integral relations (9)-(14) follow from eq. (20) by choosing $\vec{A}$ and $\vec{B}$ appropriately and integral relations (15)-(19) follow from eq. (21) by choosing Ψ and $\vec{C}$ appropriately.

This completes the proof of the lemma.

Theorem 1 : If $(p, \vec{q}, \vec{h}, \theta, \varphi)$, $p = p_r + ip_i$, $0 < \gamma < \tau/R_3$ is a non-trivial solution of eqs. (1)-(5) together with either of the boundary conditions (6)-(7), $R_T < 0$, $R_S < 0$ and $|R_S|\quad \tau|R_T|\langle 1\quad R_3\gamma/\tau\rangle$, then $p_r = 0 \Rightarrow p_i\quad 0$.

Proof: Taking $R_T\qquad |R_T|$, $R_S\qquad |R_S|$ and supposing $p_r = 0 \Rightarrow p_i = 0$ then $p = 0$, and therefore eqs. (1)-(5) become:

$$P\quad \mathrm{curl}\,\mathrm{curl}\,\vec{q}\qquad |R_T|\theta\hat{k}\quad |R_S|\phi\hat{\beta}\quad Q(\mathrm{curl}\,\vec{h})\quad \ell \tag{22}$$

$${}^2\theta\quad R_3\gamma\quad {}^2\phi\qquad \vec{q}\quad \hat{k} \tag{23}$$

and

$${}^2(\tau\phi)\qquad \vec{q}\quad \hat{k} \tag{24}$$

$$\mathrm{curl}\,\mathrm{curl}\,\vec{h} \quad \mathrm{curl}\,(\vec{q} \quad \hat{\ell}) \tag{25}$$

If $\zeta \quad \theta \quad \tau \;\; 1 \quad \frac{R_3\gamma}{\tau} \quad \phi$ then it follows from equations:

$$^2\zeta \quad 0 \tag{26}$$

Further, in view of boundary conditions (6)-(7), we have either:

$$\zeta \quad 0 \;\; \text{or} \quad\; \zeta \;\hat{n} \quad 0 \;\; \text{on} \;\; S \tag{27}$$

The only solution of equation in V subject to either of the boundary condition in eq. (27) is $\zeta = 0$. Consequently eq. (22) assumes the form:

$$P \quad \mathrm{curl}\,\mathrm{curl}\vec{q} \qquad \tau\left|\mathrm{R_T}\right| \;\; 1 \quad \frac{R_3}{\tau} \qquad \left|\mathrm{R_S}\right| \;\; \phi\hat{\beta} \quad \mathrm{Q}\,(\mathrm{curl}\,\vec{h}) \quad \hat{\ell} \tag{28}$$

Taking dot product of eq. (28) with $\vec{q}\,*$, integrating the resulting equation over the domain V and using Lemma 2, we get:

$$\underset{\mathrm{V}}{}\left|\mathrm{curl}\,\vec{q}\right|^2 \mathrm{dV} \quad Q \quad \vec{h}\,\mathrm{curl}(\vec{q} \quad \ell)\,\mathrm{dV} \quad \left|\mathrm{R_S}\right| \quad \tau\left|\mathrm{R_T}\right| \;\; 1 \quad \frac{R_3\gamma}{\tau} \quad \underset{\mathrm{V}}{}\phi(\vec{q}\,*\,\hat{\beta})\,\mathrm{dV} \tag{29}$$

Equation (29) upon using eqs. (24) and (25) and then appealing to Lemma 2 yields the equation:

$$\underset{\mathrm{V}}{}\left|\mathrm{curl}\,\vec{q}\right|^2 d\mathrm{V} \quad \mathrm{Q} \underset{\mathrm{V}}{}\left|\mathrm{curl}\,\vec{h}\right|^2 \mathrm{dV} \quad \left|\mathrm{R_S}\right| \quad \tau\left|\mathrm{R_T}\right| \;\; 1 \quad \frac{R_3\gamma}{\tau} \quad \underset{\mathrm{V}}{}\left| \;\; \phi\right|^2 \mathrm{dV} \tag{30}$$

It follows from eq. (30) that:

$$\left|\mathrm{R_S}\right| \quad \tau\left|\mathrm{R_T}\right| \;\; 1 \quad \frac{R_3\gamma}{\tau}$$

a result contrary to the given hypothesis of the theorem. Hence $p_r = 0 \Rightarrow p_i \quad 0$. This completes the proof of the theorem.

Theorem 1, in the parlance of linear stability theory, may be stated as follows. PES is not valid for the hydromagnetic GSTHC if:

$$\left|\mathrm{R_S}\right| \quad \tau\left|\mathrm{R_T}\right| \;\; 1 \quad \frac{R_3\gamma}{\tau}$$

The following corollaries are direct consequences of Theorem 1:

Cor. 1 – PES is not valid for GSTHV if $\left|\mathrm{R_S}\right| \quad \tau\left|\mathrm{R_T}\right| \;\; 1 \quad \frac{R_3\gamma}{\tau}$, Cor. 2 – PES is not valid for hydromagnetic initially top heavy GSTHC if $\tau \; 1 \; \frac{R_3 \gamma}{\tau} \; 1$, and

Cor. 3 – PES is not valid for initially top heavy GSTHC if $\tau \; 1 \; \frac{R_3\gamma}{\tau} \; 1$.

Theorem 2: If $(p, \vec{q}, \vec{h}, \theta, \varphi)$, $p \; p_r \; ip_i$, is a non trivial solution of eqs. (1)-(5) with either of the boundary conditions (6)-(7), $R_T < 0$, $R_S < 0$ and $\tau < \delta \leq 1$, then for large Q (or for large $|R_S|$ if Q = 0):

$$p_r \quad 0 \qquad p_i \quad 0$$

where
$$\delta \quad \begin{matrix} 1, \text{ if } \; R_S \; 0, \text{ and } \; Q \; 0 \\ \frac{\sigma}{\sigma_1}, \text{ if } \; R_S \; 0 \text{ and } \; Q \; 0 \end{matrix}$$

Proof: Taking $R_T \quad |R_T|$, $R_S \quad |R_S|$ and using the transformations:

$$\begin{aligned} &\tilde{\theta} \quad \frac{1 \;\; \tau}{R_3 \gamma} \theta \quad \phi \\ &\tilde{\phi} \quad \phi \\ &\tilde{\vec{q}} \quad \vec{q} \\ &\tilde{\vec{h}} \quad \vec{h} \end{aligned} \tag{31}$$

eq. (1)-(7) assume the forms:

$$\frac{p}{\sigma}\vec{q} \qquad P \quad \text{curl}\,\text{curl}\,\vec{q} \quad |R_T|\theta\hat{\beta} \quad |R_S|\phi \; \hat{\beta} \quad Q(\text{curl}\,\vec{h}) \quad \hat{\ell} \tag{32}$$

$$(\quad ^2 \quad p)\theta \qquad B\vec{q} \; \vec{\beta} \tag{33}$$

$$(\tau \quad ^2 \quad p)\phi \qquad \vec{q} \; \vec{\beta} \tag{34}$$

$$\text{curl}\,\text{curl}\,\vec{h} \quad \frac{p\sigma_1}{\sigma}\vec{h} \quad \text{curl}(\vec{q} \quad \hat{\ell}) \tag{35}$$

$$\vec{q} \quad 0, \qquad \vec{h} \quad 0 \tag{36}$$

with

$$\vec{q} \quad 0 \quad \theta \quad \phi \quad \hat{n} \quad \text{curl}\,\vec{h} \quad \text{on} \quad S \tag{37}$$

or

$$\vec{q} \quad 0 \qquad \theta \; \hat{n} \qquad \phi \; \hat{n} \quad \hat{n} \quad \mathrm{curl}\,\vec{h} \quad \text{on} \quad S \tag{38}$$

where $\mathrm{R_T} \quad R_3 \mathrm{R_T} \gamma/(1 \quad \tau)$, $\mathrm{R_S} \quad \mathrm{R_S} \quad \mathrm{R_T}, B \quad \{1 \quad (1 \quad \tau) R_3 \gamma\}$, and the sign '~' has been omitted for simplicity. Operating on eq. (32) by (δ curl curl +p) and using the vector identities:

$$\mathrm{curl}(\Psi \vec{A}) \quad \Psi \mathrm{curl}\,\vec{A} \qquad \Psi \quad \vec{A}$$

$$\mathrm{curl}(\vec{A} \quad \vec{B}) \quad (\vec{B} \qquad)\vec{A} \quad (\vec{A} \qquad)\vec{B} \quad \vec{A}\,\mathrm{div}\,\vec{B}$$

and

$$(\vec{A} \quad \vec{B}) \quad (\vec{B} \qquad)\vec{A} \quad (\vec{A} \qquad)\vec{B} \quad \vec{B}\,\mathrm{curl}\,\vec{A} \quad \vec{A} \quad \mathrm{curl}\,\vec{B}$$

with an appropriate choice of Ψ, $\vec{A}$, and $\vec{B}$, it follows that:

$$\begin{aligned} &p \;\; 1 \quad \frac{\delta}{\sigma} \quad \mathrm{curl\,curl}\,\vec{q} \quad \frac{p^2}{\sigma}\vec{q} \quad p \quad P \\ &|\mathrm{R_S}|\{\delta[\quad (\mathrm{div}\,\phi\hat{\beta}) \qquad {}^2\phi\hat{\beta}] \quad p\phi\hat{\beta}\} \\ &|\mathrm{R_T}|\{\delta[\quad (\mathrm{div}\,\theta\hat{\beta}) \qquad {}^2\theta\hat{\beta}]\} \\ &+\mathrm{Q}\{\delta[\hat{\ell} \quad \mathrm{curl\,curl\,curl}\,\vec{h} \qquad (\hat{\ell} \quad \mathrm{curl\,curl}\,\vec{h}) \quad p(\mathrm{curl}\,\vec{h}) \quad \hat{\ell}]\} \\ &= \quad \delta\,\mathrm{curl\,curl\,curl\,curl}\,\vec{q} \end{aligned} \tag{39}$$

Taking the dot product of eq. (39) with $\vec{q}\,*$, integrating the resulting equation over the domain V and Lemma 2, we have:

$$\begin{aligned} &p \;\; 1 \quad \frac{\delta}{\sigma} \; \underset{\mathrm{V}}{} |\mathrm{curl}\,\vec{q}|^2 \quad \frac{p^2}{\sigma} \; \underset{\mathrm{V}}{} |\vec{q}|^2 \mathrm{dV} \quad |\mathrm{R_T}| \underset{\mathrm{V}}{} (\delta \quad {}^2\theta \quad p\theta)(q * \hat{\beta})\mathrm{dV} \\ &|\mathrm{R_S}| \underset{\mathrm{V}}{} (\delta \quad {}^2\phi \quad p\phi)(\vec{q} * \hat{\beta})\mathrm{dV} \quad p\mathrm{Q} \underset{\mathrm{V}}{} \vec{h}\,\mathrm{curl}(\vec{q} \quad \hat{\ell})\mathrm{dV} + \\ &+\mathrm{Q}\delta \underset{\mathrm{V}}{} \mathrm{curl\,curl}\,\vec{h}\,\mathrm{curl}(\vec{q} \quad \hat{\ell})\,\mathrm{dv} = \\ &= \quad \delta \underset{\mathrm{V}}{} \vec{q}\,\mathrm{curl\,curl\,curl\,curl}\,\vec{q}\mathrm{dV} \end{aligned} \tag{40}$$

Since Q (the ratio of magnetic to viscous forces) is very large, the effect of viscosity is thus significant near the bounding surfaces and in the above equation the integral on the right hand side (resulting from the viscous forces) is negligible in comparison with the last integral on the left hand side (resulting from the magnetic force) [6]. Consequently, taking the right hand side of eq. (40) to zero, eliminating $(\vec{q} * \beta *)$ and $(\vec{q} * \quad \ell *)$ from the resulting equation by using eqs. (33)-(35) and then appealing to Lemma 2, we get:

$$p \;\; 1 \;\; \frac{\delta}{\sigma} \;\; \underset{V}{} \left|\mathrm{curl}\,\vec{q}\right|^2 \mathrm{dV} \;\; \frac{p^2}{\sigma} \underset{V}{} \left|\vec{q}\right|^2 d\mathrm{V} \;\; \left|\mathrm{R_T}\right| \underset{V}{} [\delta(\;\; {}^2\theta)^2 \;\; |p|^2|\theta|^2]\,\mathrm{dV} \;\; \left|\mathrm{R_T}\right|(p^*\delta \;\; p)$$

$$\underset{V}{}\left| \;\; \theta\right|^2 \mathrm{dV} \;\; \left|\mathrm{R_S}\right| \underset{V}{} \delta \;\; \left| \;\; {}^2\theta\right|^2 \;\; |p|^2|\phi|^2 \;\; \mathrm{dV} \;\; \left|\mathrm{R_S}\right|(p^*\delta \;\; \tau p) \underset{V}{}\left| \;\; \phi\right|^2 \mathrm{dV} +$$

$$+\mathrm{Q} \underset{V}{} \;\; \delta\left|\mathrm{curl\;curl}\;\vec{h}\right|^2 \;\; \frac{|p|^2\sigma_1}{\sigma}\left|\vec{h}\right|^2 \;\; \mathrm{dV} +$$

$$+\mathrm{Q} \;\; \frac{p^*\delta\sigma_1}{\sigma} \;\; p \underset{V}{} \left|\mathrm{curl}\;\vec{h}\right|^2 \mathrm{dV} \;\; 0 \tag{41}$$

Equating the imaginary past of eq. (41) to zero and assuming $p_i \;\; 0$, we get:

$$1 \;\; \frac{\delta}{\sigma} \underset{V}{} \left|\mathrm{curl}\,\vec{q}\right|^2 \mathrm{dV} \;\; \frac{2p_r}{\sigma} \underset{V}{} \left|\vec{q}\right|^2 \mathrm{dV} \;\; \left|\mathrm{R_T}\right|(1 \;\; \delta) \underset{V}{}\left| \;\; \theta\right|^2 \mathrm{dV} +$$

$$+\left|\mathrm{R_S}\right|(\delta \;\; \tau) \underset{V}{}\left| \;\; \phi\right|^2 \mathrm{dV} \;\; 0 \tag{42}$$

Equation (42) cannot obviously be satisfied under the conditions of the theorem. Hence we must have $p_i = 0$

This comples the proof of the theorem.

Theorem 2 implies that the hydromagnetic GSTHC on arbitrary neutral or unstable mode is definitely non-oscillatory in character and in particular PES is valid if $\tau\sigma_1 \;\; \sigma \;\; \sigma_1$. Further, this theorem also implies the validity of this result for the GSTHC if $\tau < 1$.

Theorem 3: If $(p,\vec{q},\theta,\phi,\vec{h}), p = p_r + ip_i, \; p_r \;\; 0, p_i \;\; 0$ is a non-trivial solution of eqs. (1)-(5) together with the boundary conditions (6) and $\mathrm{R_T} < 0$, $\mathrm{R_S} < 0$, and $\delta > 1$ then for large Q (or for large $\left|\mathrm{R_S}\right|$ if Q = 0):

$$|p| \;\; \hat{\delta}[|R_T|(\delta \;\; 1)B^2 \;\; |R_S|]$$

where $\hat{\delta} \;\; \ell^2\sigma/\lambda(\sigma \;\; \delta), \delta$ is as in Theorem 2 and ℓ and λ are as in Lemma 1.

Proof: It follows from eq. (23) that:

$$\underset{V}{}(\;\; {}^2\theta \;\; p\theta)(\;\; {}^2\theta^* \;\; \rho^*\theta^*)\mathrm{dV} \;\; B^2 \underset{V}{}\left|\vec{q} \;\; \hat{\beta}\right|^2 \mathrm{dV} \tag{43}$$

Equation (43) upon using Lemma 2 gives:

$$\underset{V}{}\left| \;\; {}^2\theta\right|^2 \mathrm{dV} \;\; 2p_r \underset{V}{}\left| \;\; \theta\right|^2 \mathrm{dV} \;\; |p|^2 \underset{V}{}|\theta|^2 \mathrm{dV} \;\; \underset{V}{}\left|\vec{q} \;\; \hat{\beta}\right|^2 \mathrm{dV} \tag{44}$$

Equation (44), upon using p_r 0, p_i 0 give:

$$ {}_{V}|\theta|^2 \mathrm{dV} \quad \frac{B^2}{|p|^2} {}_{V}|\vec{q} \;\; \hat{\beta}|^2 \mathrm{dV} \quad \frac{B^2}{|p|^2} {}_{V}|\vec{q}|^2 \mathrm{dV} \tag{45}$$

Again multiplying eq. (33) by θ^*, integrating over the domain V, using Lemma 2 and equating the real parts of the resulting equation, we have:

$$ {}_{V}| \;\; \theta|^2 \mathrm{dV} \quad p_r \; {}_{V}|\theta|^2 \mathrm{dV} \quad \text{Real part of} \quad B \; {}_{V}\theta^* |\vec{q} \;\; \hat{\beta}| \mathrm{dV} $$

$$ \left| \mathrm{B} \; {}_{V}\theta^* (\vec{q} \;\; \hat{\beta}) \mathrm{dV} \right| \quad B \; {}_{V}|\theta||\vec{q} \;\; \hat{\beta}| \mathrm{dV} $$

which upon using Schwartz's inequality and the fact that p_r 0, gives:

$$ {}_{V}| \;\; \theta|^2 \mathrm{dV} \quad B \quad {}_{V}|\theta|^2 \mathrm{dV}^{\;½} \quad B \quad {}_{V}|\vec{q} \;\; \hat{k}|^2 \mathrm{dV}^{\;½} $$

$$ B \quad {}_{V}|\theta|^2 \mathrm{dV}^{\;½} \quad B \quad {}_{V}|\vec{q}|^2 \mathrm{dV}^{\;½} \tag{46}$$

Combining inequalities (45) and (46), we get:

$$ {}_{V}| \;\; \theta|^2 \mathrm{dV} \quad \frac{B \;\; B}{|p|} {}_{V}|\vec{q}|^2 \mathrm{dV} \tag{47}$$

Further, the solenoidal character of the velocity field $\vec{q}$ namely div $\vec{q}$ 0, implies that:

$$ {}_{V}|\mathrm{curl}\,\vec{q}|^2 \mathrm{dV} \quad {}_{V}(\vec{q}\,^* \mathrm{curl}\,\mathrm{curl}\vec{q}) \mathrm{dV} \quad {}_{V}\vec{q}\,^* \;\; {}^2\vec{q} \, \mathrm{dV} $$

which upon taking $\vec{q}$ (u, v, w) gives:

$$ {}_{V}|\mathrm{curl}\vec{q}|^2 \mathrm{dV} \quad {}_{V}(| \;\; u|^2 \;\; | \;\; v|^2 \;\; | \;\; w|^2) \mathrm{dV} \tag{48}$$

Equation (48) together with Lemma 1 yields the inequality:

$$ {}_{V}|\vec{q}| \mathrm{dV} \quad \frac{\ell^2}{\lambda} {}_{V}|\mathrm{curl}\,\vec{q}|^2 \mathrm{dV} \tag{49}$$

Inequalities (47) and (49) implies that:

$$\int_V |\theta|^2 \mathrm{dV} \quad \frac{B^2\ell^2}{\lambda|p|}\int_V |\mathrm{curl}\,\vec{q}|^2 \mathrm{dV} \tag{50}$$

Similarly proceeding from eq. (34), and emulating the steps in the derivation of inequality (50), we have:

$$\int_V |\phi|^2 \mathrm{dV} \quad \frac{\ell^2}{\lambda\tau|p|}\int_V |\mathrm{curl}\;|^2 \mathrm{dV} \tag{51}$$

Using inequalities (5) and (51) in eq. (42), we get:

$$\frac{\sigma \quad \delta}{\sigma}\{p| \quad \hat{\delta}\langle|\mathrm{R_T}|(\delta \quad 1)B^2 \quad |\mathrm{R_S}|\rangle\} \int_V |\mathrm{curl}\,\vec{q}|^2 \mathrm{dV}$$
$$\frac{2p\mathrm{e}}{\sigma}\int_V |\vec{q}|^2 \mathrm{dV} \quad \delta|\mathrm{R_S}| \int_V |\phi|^2 \mathrm{dV} \quad 0 \tag{52}$$

Inequalities (52) clearly implies that:

$$|p|\langle\hat{\delta}[|\mathrm{R_T}|(\delta \quad 1)B^2 \quad |\mathrm{R_S}|]$$

This completes the proof of the theorem.

Theorem 3 implies that the complex growth rate of an arbitrary oscillatory perturbation which may be neutral or unstable for the hydromagnetic GSTHC lies inside a semi-circle with centre origin and:

$$\text{Radius} \quad \delta[|\mathrm{R_T}|(\delta \quad 1)B^2 \quad |\mathrm{R_S}|], \quad \delta \quad \frac{\sigma}{\sigma_1}$$

in the right half of the complex *p*-plane.

Conclusion

The present paper investigates the instability of Dufour-driven thermosolutal convection of a fluid completely confined in an arbitrary region bounded by rigid walls in the presence of a uniform magnetic field applied in an arbitrary direction. It has been found that Principle of exchange of stabilities is not valid for the hydromagnetic generalized Stern's thermohaline configuration if $|\mathrm{R_S}| \quad \tau|\mathrm{R_T}|[1 \quad (R_3\gamma/\tau)]$. Secondly, for large Chandrasekhar number and $\tau < \delta < 1$, a neutral or unstable mode is definitely non oscillatory in character and in particular PES is valid. Finally the complex growth rate of an arbitrary oscillatory perturbation which may be neutral or unstable lies inside a semi-circle with centre origin and radius = $\hat{\delta}[|\mathrm{R_T}|(\delta \quad 1)B^2 \quad |\mathrm{R_S}|]$ in the right half of the complex p-plane. Further, the results for the horizontal layer geometry in case of single-diffusive or double diffusive fluids follows as a consequence by taking $\lambda = \pi^2$, $l = 1$ respectively.

Nomenclature

d – depth of layer, [m]
g – acceleration due to gravity, [m/s^2]
h – magnetic field, [Gs]
P – pressure, [Pa]
p – growth rate, [1/s]
Q – Chandrasekhar number, [–]
$\vec{q}$ – velocity, [m/s]
R_S – solutal Rayleigh number, [–]
R_T – thermal Rayleigh number, [–]
R_3 – gradient ratio, (= β'/β), [–]
t – time, [s]

Greek letters

α – coefficient of thermal expansion, [1/K]
α' – coefficient of solute expansion, [1/K]
β – uniform temperature gradient, [K/m]
β' – uniform concentration gradient, [K/m]
Γ – ratio of two Rayleigh numbers, [–]
γ – Dufour number, [–]
δ – ratio of two Prandtl numbers, [–]
η – electrical resistivity, [m^2/s]
θ – perturbation in temperature, [K]
κ – thermal diffusivity, [m^2/s]
κ' – mass diffusivity, [m^2/s]
ν – kinematic viscosity, [m^2/s]
σ – Prandtl number, (= ν/κ), [–]
σ_1 – magnetic Prandtl number, (= ν/η), [–]
ρ – density, [kg/m^3]
τ – Lewis number, (= κ'/κ), [–]
ϕ – perturbation in concentration, [Kg]

References

[1] Fitts, D. D., Nonequlibrium Thermodynamics, McGraw Hill, New York, 1962
[2] De Groot, S. R., Mazur, P., Nonequilibrium Thermodynamics, North Holland, Amsterdam, 1962
[3] Mohan, H., Dufour-Driven Thermosolutal Convection of the Veronis Type, *J. Math. Anal. Appl., 218* (1998), pp. 569-580
[4] Veronis, G., On Finite Amplitude Instability in Thermohaline Convection, *J. Marine Res. 23* (1965), pp. 1-17
[5] Banerjee, M. B., Katoch, D. C. R., Dube, G. S., Banerjee, K., Bounds for Growth Rate of a Perturbation in Thermohaline Convection, Proc. Roy. Soc. London, Ser. *A 378* (1981), pp. 301-304
[6] Sherman, M., Ostrach, S., On the Principle of Exchange of Stability for the Magnetohydrodynamic Thermal Stability Problem Incompletely Confined Fluids, *J. Fluid Mech., 23* (1966), pp. 661-671

[7] Stern, M. E., The Salt Fountain and Thermohaline Convection, *Tellus. 12* (1960), pp. 172-175
[8] Joseph, D. D., Stability of Fluid Motion I, Springer Verlag, New York, 1976

Authors' addresses:

H. Mohan, P. Kumar, P. Singh
Department of Mathematics, ICDEOL
Himachal Pradesh University
Summer Hill, SHIMLA-5, India

Corresponding author (H. Mohan):
E-mail: hms_math_hpu@rediffmail.com

Paper submitted: October 24, 2004
Paper revised: February 15, 2005
Paper accepted: April 6, 2005