

# ON THE EVOLUTION OF LAMINAR TO TURBULENT TRANSITION AND BREAKDOWN TO TURBULENCE

by

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*Starting from the basic conservation laws of fluid flow, we investigated transition and breakdown to turbulence of a laminar flat plate boundary layer exposed to small, statistically stationary, two-component, three-dimensional disturbances. The derived equations for the statistical properties of the disturbances are closed using the two-point correlation technique and invariant theory. By considering the equilibrium solutions of the modeled equations, the transition criterion is formulated in terms of a Reynolds number based on the intensity and the length scale of the disturbances. The deduced transition criterion determines conditions that guarantee maintenance of the local equilibrium between the production and the viscous dissipation of the disturbances and therefore the laminar flow regime in the flat plate boundary layer. The experimental and numerical databases for fully developed turbulent channel and pipe flows at different Reynolds numbers were utilized to demonstrate the validity of the derived transition criterion for the estimation of the onset of turbulence in wall-bounded flows.*

*Key words: transition to turbulence, two-point correlations, closure problem, transition criterion*

## **Introduction**

There have been many theoretical and experimental studies of the processes that cause laminar to turbulent transition in boundary layers and the phenomena occurring in the inner region of fully developed turbulent flows. These studies led to conclusion that there is a strong similarity between the mechanisms responsible for transition and the continuous production of turbulence close to the solid boundaries. Hinze <sup>1</sup> provided a brief review of the subject based on linear and non-linear stability theory. Hinze also presented experimental results deduced from hot-wire measurements and flow visualizations which revealed interesting details of the sequence of the events during the transition process. Studies of the dynamics of coherent structures close to the wall by Kline *et al.*, <sup>2</sup>, Kim, Kline, and Reynolds <sup>3</sup>, and Falco <sup>4</sup> showed remarkable analogies to the sequence of events leading to transition which led Laufer <sup>5, 6</sup> to the

conclusion that these processes are very closely interconnected and therefore should be treated theoretically using the same mathematical concepts. This issue was raised by J. Laufer, P. Klebanoff, R. Falco, and M. Landahl during participation in the round-table discussion organized by Z. Zarić at the ICHMT Symposium on Structure of Turbulence in Heat and Mass Transfer (Dubrovnik, 1980). Though there has been an explosion of activity during last two decades, apparently no-one has yet succeeded in providing a description of transition and breakdown to turbulence using statistical techniques.

The purpose of this paper is to fill the gaps in the theoretical treatment of the transition process using statistical techniques. An attempt is made to establish quantitative links between the transition process and fully developed turbulence using stochastic tools suitable for describing random, three-dimensional flow fields. We shall provide rational approximations for the mechanisms involved in the transition process using the two-point correlation technique and invariant theory and finish with a closed set of transport equations which are, however, identical with those describing the behaviour of fully developed turbulent flow in the region of the viscous sublayer close to the wall. It is expected that these model equations will serve as a starting point for developing methods for the prediction of transitional and fully developed turbulent flows with the same set of transport equations.

### Basic equations

Starting from the Navier-Stokes and the continuity equations for a viscous incompressible fluid:

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k}, \quad \frac{\partial u_k}{\partial x_k} = 0, \quad i, k = 1, 2, 3 \quad (1)$$

and introducing the conventional method of separating the instantaneous velocity  $u_i$ , and the pressure  $p$  into the mean-laminar flow and disturbances  $u'_i$ , and  $p'$  superimposed on it:

$$u_i = U_i + u'_i, \quad p = P + p' \quad (2)$$

one obtains the equations for the disturbances:

$$\frac{\partial u'_i}{\partial t} + U_k \frac{\partial u'_i}{\partial x_k} - u'_k \frac{\partial U_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u'_i}{\partial x_k \partial x_k}, \quad \frac{\partial u'_k}{\partial x_k} = 0 \quad (3)$$

In the derivation of the above equations, it is assumed that the disturbances are much smaller than the corresponding quantities of the mean flow:

$$u'_i \ll U_i, \quad p' \ll P \quad (4)$$

and that they satisfy the Navier-Stokes and the continuity equations. By systematic manipulation of eq. (3) it is possible to obtain transport equations for the “apparent” turbulent stresses (see, for example, 1, pp. 323-324):

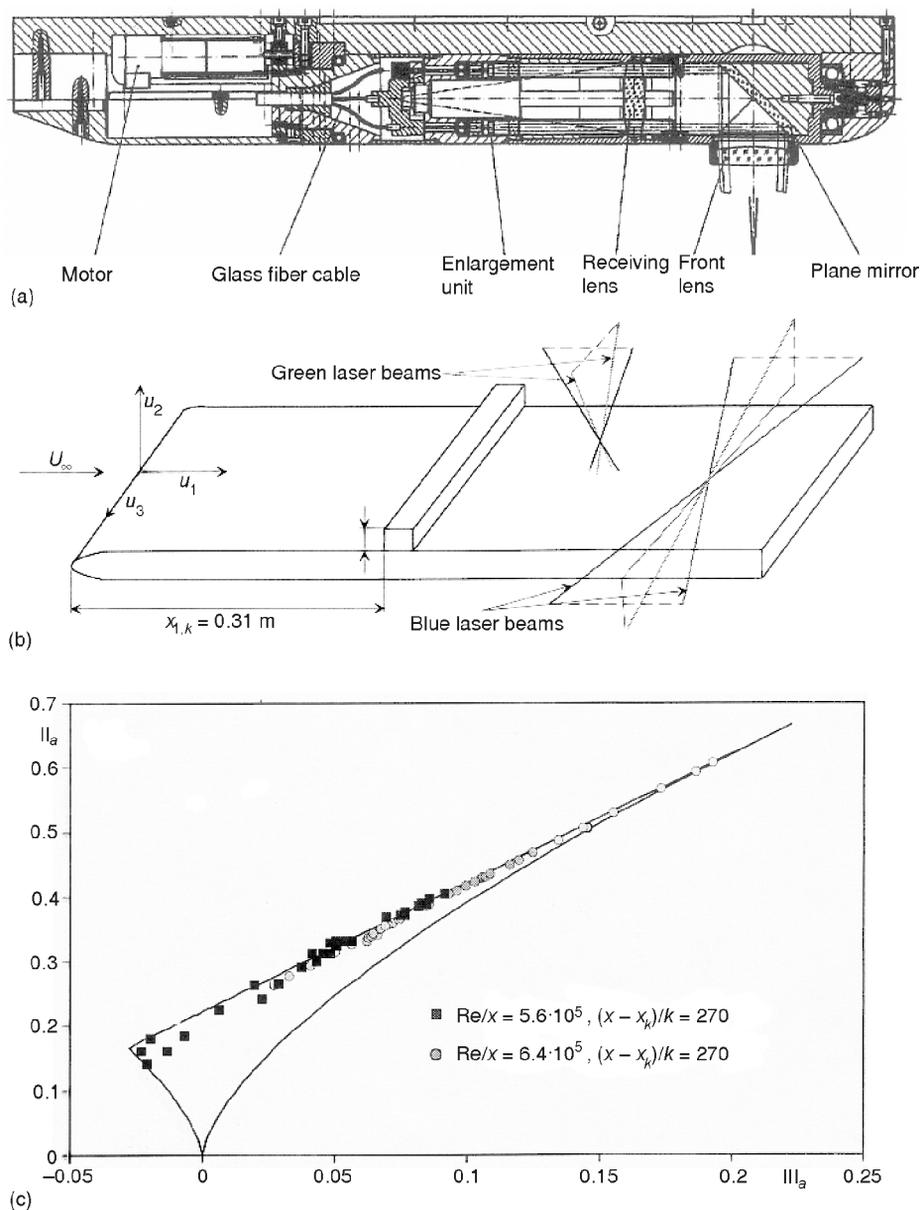
$$\begin{aligned} & \frac{\partial \overline{u_i u_j}}{\partial t} - U_k \frac{\partial \overline{u_i u_j}}{\partial x_k} - \overline{u_j u_k} \frac{\partial U_i}{\partial x_k} - \overline{u_i u_k} \frac{\partial U_j}{\partial x_k} \\ & \underbrace{\frac{1}{\rho} \overline{u_j} \frac{\partial p}{\partial x_i} - \overline{u_i} \frac{\partial p}{\partial x_j}}_{\Pi_{ij}} - \underbrace{2\nu \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_j}}{\partial x_k}}_{\varepsilon_{ij}} - \nu \frac{\partial^2 \overline{u_i u_j}}{\partial x_k \partial x_k} \end{aligned} \quad (5)$$

In the above equations one can identify two different types of unknown correlations: the velocity/pressure gradient correlations  $\Pi_{ij}$  and the dissipation correlations  $\varepsilon_{ij}$ . These correlations must be expressed in terms of  $U_i$  and  $\overline{u_i u_j}$  in order to close the resultant eq. (5) for the stresses.

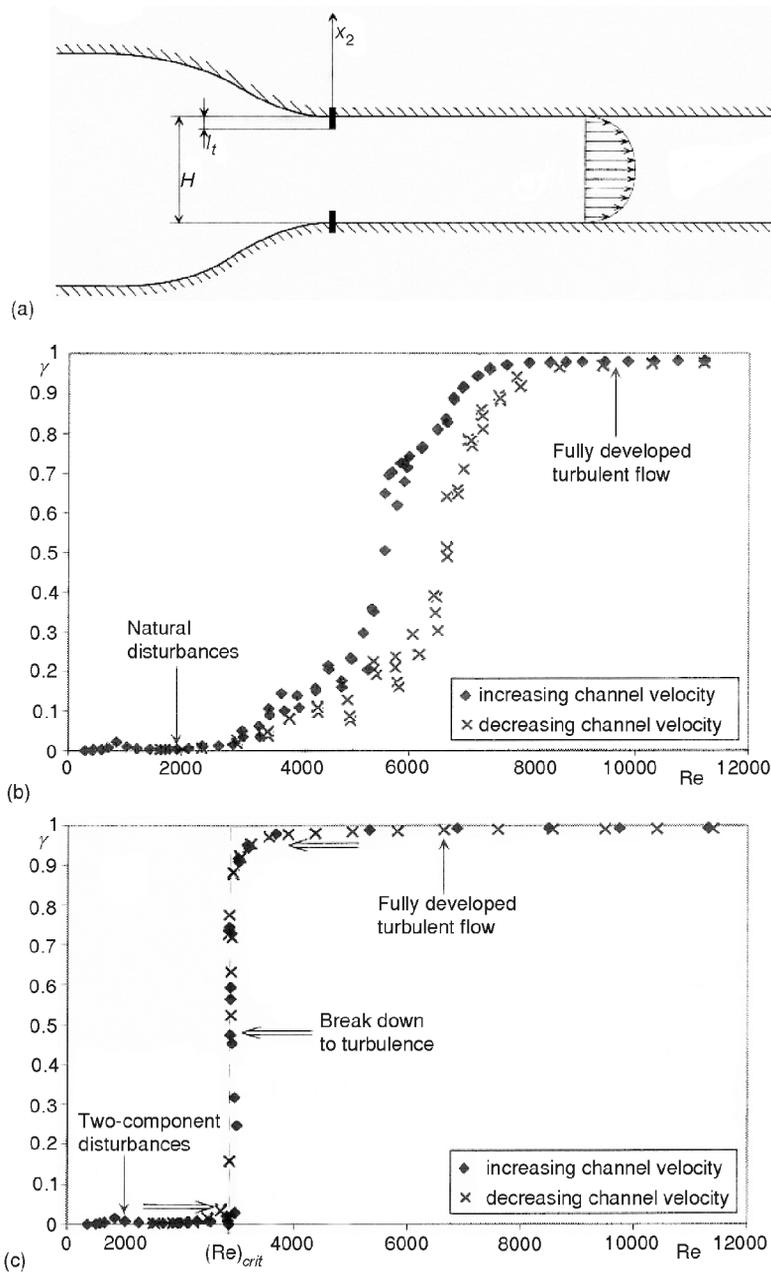
### The closure problem for small, two-component, three-dimensional disturbances

Let small disturbances be statistically stationary, two-component and three-dimensional consisting of longitudinal,  $u_1$ , and transverse,  $u_2$ , velocity pulsations, while the lateral component,  $u_3$ , of the pulsations is assumed to be zero everywhere in time and space. Such disturbances can be produced by a small, two-dimensional roughness element placed at the wall in a laminar boundary layer as shown in fig. 1. Measurements of the anisotropy invariants  $\Pi_a$  and  $\text{III}_a$ , to be discussed in subsection to follow, of the disturbances clearly demonstrate that these are almost two-component for different free stream velocities and low Reynolds numbers. The laminar to turbulence transition process initialized by the two-component disturbances is characterized by abrupt and explosive breakdown to the fully developed turbulent state at a fixed value of the Reynolds number which is very close those met in engineering practice. This is indicated in fig. 2(c), from which it appears that approaching the break downpoint from the laminar, two-component state and from the fully developed turbulent state results in the same estimate of the critical Reynolds number. We shall exploit this experimental evidence in order to provide the quantitative link between results of the theoretical considerations of the transition process with those of fully developed turbulence which are available from numerous experiments and also from direct numerical simulations.

A few remarks follow with respect to the role of the natural disturbances in the transition process. Figure 2(b) shows that for such disturbances laminar to turbulence transition is weaker than transition induced by the two-component disturbances shown in fig. 2(c). Natural disturbances, which originate in technical practice, can be considered in a statistical sense of fig. 3 as nearly axisymmetric. For this type of disturbances theoretical considerations, following same analytical path as used in this study, show that



**Figure 1. Anisotropy-invariant mapping of the disturbances generated by small, two-dimensional roughness element in a initially laminar flat plate boundary layer from [7]**  
 (a) specially designed two-component laser-Doppler system for near-wall measurements; (b) schematic of flat plate arrangement in the wind tunnel with layout of two different beam configurations which allowed measurements of all components of the apparent stresses of the disturbances; (c) traces of the joint variations of invariants  $II_a$  and  $III_a$  across the anisotropy invariant map confirm the two-component nature of the disturbances



**Figure 2.** Intermittency ( $\gamma$ ) measurements of the transition process from laminar to turbulent states at the channel centreline from Fischer 8 : (a) channel flow test section; (b) transition due to the natural disturbances is accompanied by large hysteresis in the experimental data; (c) transition due to two-component disturbances

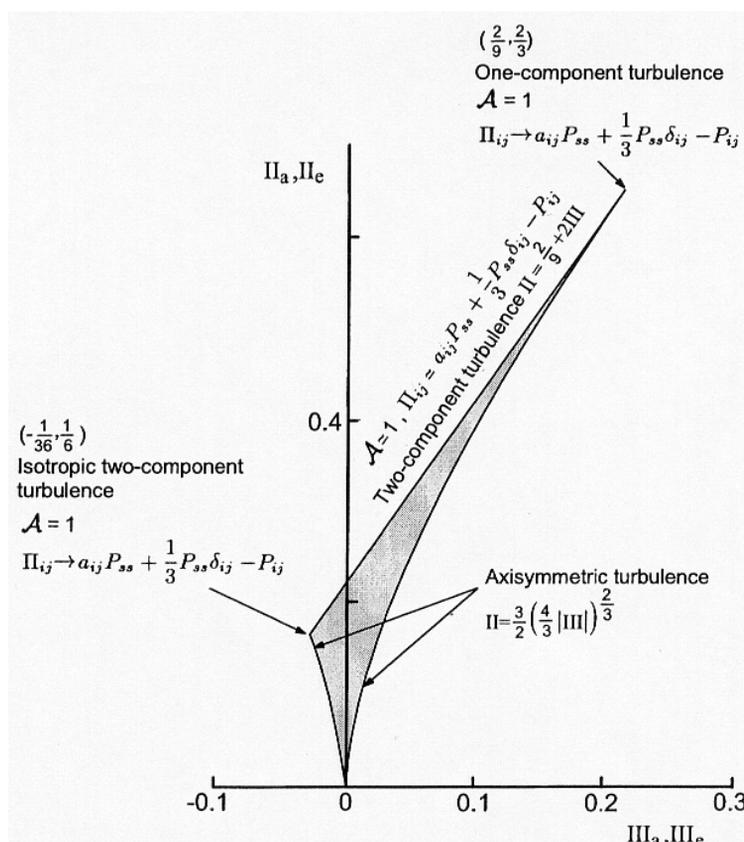


Figure 3. Anisotropy invariant map and the asymptotic forms for the unknown correlations involved in the equations for the apparent stresses

the critical Reynolds number for breakdown to turbulence depends strongly on the anisotropy of the disturbances. If the intensity of the streamwise velocity component is smaller than the intensities of the normal and lateral components,  $III_a < 0$ , the transition process is promoted and occurs at low Reynolds numbers. For the reverse situation when the intensity of the streamwise velocity component is larger than the intensities of the normal and lateral components,  $III_a > 0$ , the transition process is delayed and occurs at high Reynolds number. Since the continuity equation near the wall dictates that  $III_a > 0$  the results in fig. 2(b) are logical and not surprising.

For small, statistically stationary, two-component, three-dimensional disturbances it is possible to attack the closure of eq. (5) using the two-point correlation technique developed by Chou 9 and invariant theory introduced by Lumley and Newman 10 . Application of the two-point correlation technique permits the separation of the inhomogeneous effects in the treatment of the unknown terms involved in eq. (5) and recasting of the inhomogeneous problem into the corresponding problem of a statistically

homogeneous flow field. Then, using invariant theory, it is possible to isolate the effects of the anisotropy in the turbulent stresses from all other flow properties, which allows rational construction of the closure approximations that include all physically realistic states of the disturbances.

*Application of the two-point correlation technique for interpretation of  $\varepsilon_{ij}$*

Let us first consider closure for the terms which are related to the dissipation process:

$$\varepsilon_{ij} = \overline{v \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} \quad (6)$$

that appear in eq. (5). The most efficient procedure to treat these correlations is based on the two-point correlation technique that was originally developed by Chou 9 and subsequently refined by Kolovandin and Vatutin 12, and Jovanović, Ye, and Durst 13. However, application of this technique to the study of the dynamics of the disturbances is complicated, tedious and very demanding for the reader. Here we shall provide only a brief account of the parts of the subject which are relevant for the present study.

In order to separate the effect of local character from global, large-scale fluid motion, we must first express the dissipation correlations  $\varepsilon_{ij}$  in a coordinate system relative to two closely separated points A and B in space as follows:

$$\begin{aligned} \varepsilon_{ij} &= \overline{v \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} = v \lim_{A \rightarrow B} \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} \\ &= v \lim_{A \rightarrow B} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} \overline{(u_i)_A (u_j)_B} \end{aligned} \quad (7)$$

Expressing the partial differential operators in eq. (7) at points A and B as functions of the position in space and the relative coordinates between these two points:

$$\xi_k = (x_k)_B - (x_k)_A \quad (8)$$

and taking the limit  $A \rightarrow B$  yields 13

$$\varepsilon_{ij} = \overline{v \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} = \underbrace{\frac{1}{4} v \Delta_x \overline{u_i u_j}}_{\text{inhomogeneous}} + \underbrace{v (\Delta_\xi \overline{u_i u_j})_0}_{\text{homogeneous}} \quad (9)$$

where the double prime, ( $''$ ), indicates a value of the two-point correlation function at point B,  $\overline{(u_i)_A(u_j)_B} = \overline{u_i u_j}$ , the subscript zero ( $_0$ ) represents zero relative separation in space,  $\bar{\xi} = 0$ , and  $\Delta$  corresponds to the Laplace operator ( $\Delta_x = \partial^2 / \partial x_k \partial x_k$ ,  $\Delta_\xi = \partial^2 / \partial \xi_k \partial \xi_k$ ).

Equation (9) shows that  $\varepsilon_{ij}$  is composed of an inhomogeneous part  $(1/4)v\Delta_x \overline{u_i u_j}$  and a homogeneous part  $\varepsilon_{ij}^h = v(\Delta_\xi \overline{u_i u_j})_0$ . Since the tensor  $\varepsilon_{ij}$  is symmetrical, from eq. (9) it follows that:

$$(\Delta_\xi \overline{u_i u_j})_0 = (\Delta_\xi \overline{u_j u_i})_0 \tag{10}$$

the two-point velocity correlation of second rank in the limit when  $\bar{\xi} = 0$  satisfies the same relationship as in a statistically homogeneous flow field. This peculiarity of the two-point velocity correlation, deduced only from kinematic considerations, permits us to introduce the concept of local homogeneity for the disturbances, which leads to radical simplifications of the dynamic equations for the dissipation correlations.

Since the inhomogeneous part of  $\varepsilon_{ij}$  can be directly related to  $\overline{u_i u_j}$  we need to consider only the homogeneous part of eq. (9). Using the two-point correlation technique, kinematic constraints and the continuity equation, it can be shown [9, 12, 14] that the components of the homogeneous part of eq. (9) can be interpreted analytically in terms of its trace  $\varepsilon_h = v(\Delta_\xi \overline{u_s u_s})_0$  and the turbulent stresses  $\overline{u_i u_j}$ . Therefore, only the equation for  $\varepsilon_h$  needs to be considered. This equation is obtained by operating the dynamic equation for the two-point velocity correlation in a relative coordinate system with respect to  $-v\Delta_\xi$  and setting  $\bar{\xi} = 0$  to obtain [13]:

$$v \frac{\partial}{\partial t} (\Delta_\xi \overline{u_s u_s})_0 - v U_k \frac{\partial}{\partial x_k} (\Delta_\xi \overline{u_s u_s})_0 - 2v (\Delta_\xi \overline{u_k u_s})_0 \frac{\partial U_s}{\partial x_k} + 2v \frac{\partial^2}{\partial \xi_i \partial \xi_k} \overline{u_s u_s} - \frac{\partial U_k}{\partial x_i} - 2v^2 (\Delta_\xi \Delta_\xi \overline{u_s u_s})_0 - \frac{1}{2} v^2 \Delta_x (\Delta_\xi \overline{u_s u_s})_0 \tag{11}$$

The approximate equation for the homogeneous part of the dissipation rate involves only the derivatives of two-point velocity correlations. In the derivation of this equation the concept of local homogeneity was utilized by applying the relationships for the derivatives of the two-point correlation functions for zero separation ( $\bar{\xi} = 0$ ) that are valid in a statistically homogeneous flow field.

The first two terms on the right-hand side of eq. (11) are the production terms that originate from the mean velocity gradient. A firm analytical closure for these terms can be formulated only for the case of axisymmetric disturbances. For such disturbances Jovanović, Otić, and Bradshaw [15] showed that the above-mentioned terms are equal, and their sum is given by:

$$2\nu(\Delta_{\xi}\overline{u_k u_s})_0 \frac{\partial U_s}{\partial x_k} - 2\nu \frac{\partial^2}{\partial \xi_i \partial \xi_k} \overline{u_s u_s} \frac{\partial U_k}{\partial x_i} - 2A \frac{\varepsilon_h \overline{u_i u_k}}{k} \frac{\partial U_i}{\partial x_k} \quad (12)$$

where  $k = 1/2\overline{u_k u_k} = 1/2q^2$  and the scalar function  $A$  depends on the anisotropy in  $\overline{u_i u_j}$  and  $\varepsilon_{ij}^h$ , to be discussed later. Figure 1 shows that, for the limiting states of axisymmetric disturbances which lie at the two-component limit,  $A = 1$  and therefore we may suggest closure for such a disturbance state in the following form:

$$2\nu(\Delta_{\xi}\overline{u_k u_s})_0 \frac{\partial U_s}{\partial x_k} - 2\nu \frac{\partial^2}{\partial \xi_i \partial \xi_k} \overline{u_s u_s} \frac{\partial U_k}{\partial x_i} - 2 \frac{\varepsilon_h \overline{u_i u_k}}{k} \frac{\partial U_i}{\partial x_k} \quad (13)$$

The third term on the right-hand side of eq. (11) represents the viscous destruction of  $\varepsilon_h$  and can be approximated using the scaling arguments based on the asymptotic balance of the dissipation rate equation 16 in the form:

$$2\nu^2(\Delta_{\xi}\Delta_{\xi}\overline{u_s u_s})_0 = 2\nu^2 \frac{q^2}{\lambda^2 \eta^2} \quad (14)$$

where  $\lambda$  and  $\eta$  are the Taylor's microscale and the Kolomogorov's length scale, respectively. These scales can be interpreted in terms of the dissipation rate as:

$$\lambda = \frac{5\nu q^2}{\varepsilon_h}^{1/2} \quad \text{and} \quad \eta = \frac{\nu^3}{\varepsilon_h}^{1/4} \quad (15)$$

Using the above relations (14) can be written as:

$$2\nu^2(\Delta_{\xi}\Delta_{\xi}\overline{u_s u_s})_0 = \frac{\sqrt{5}}{25} R_{\lambda} \frac{\varepsilon_h^2}{k} \quad (16)$$

where  $R_{\lambda} = q\lambda/\nu$  is the Reynolds number based on statistical properties of the disturbances. The above result also follows from the asymptotic balance of the dissipation rate eq. (11) at the wall where the viscous destruction  $2\nu^2(\Delta_{\xi}\Delta_{\xi}\overline{u_s u_s})_0$  is in balance with the viscous diffusion  $(1/2)\nu^2\Delta_x(\Delta_{\xi}\overline{u_s u_s})_0$  17 . It is therefore not surprising that both of the suggested proposals (13) and (16) for the closure of (11) agree closely with the available data from direct numerical simulations of wall-bounded flows in the region of the viscous sublayer 18 .

### *Construction of the closure approximations using invariant theory*

We shall now apply invariant theory developed by Lumley and Newman 10 to formulate the closures for partition of the homogeneous part of the dissipation tensor

and also for the velocity/pressure gradient correlations. These authors introduced the tensor:

$$a_{ij} = \frac{\overline{u_i u_j}}{q^2} = \frac{1}{3} \delta_{ij} \quad (17)$$

and its scalar invariants:

$$\Pi_a = a_{ij} a_{ji} \quad \text{and} \quad \text{III}_a = a_{ij} a_{jk} a_{ki} \quad (18)$$

to quantify the anisotropy and define the limiting states of the disturbances. A cross plot of  $\Pi_a$  versus  $\text{III}_a$  for axisymmetric disturbances,  $\Pi_a = (3/2) (4/3) |\text{III}_a|^{2/3}$ , and two-component disturbances,  $\Pi_a = 2/9 + 2\text{III}_a$ , defines the anisotropy invariant map which, according to Lumley [11], bounds all physically realizable disturbances. This plot is shown in fig. 3 and the asymptotic forms for the unknown correlations involved in eq. (5) that can be derived for the two-component disturbances.

For the axisymmetric disturbances, Jovanović and Otić [19] showed that all second-rank correlation tensors involved in eq. (5) are linearly aligned in terms of each other. For such disturbances we may write:

$$e_{ij} = A a_{ij}, \quad A = \left( \frac{\Pi_e}{\Pi_a} \right)^{1/2} \quad (19)$$

where  $e_{ij}$  is the anisotropy tensor of the homogeneous part of  $\varepsilon_{ij}$ :

$$e_{ij} = \frac{\varepsilon_{ij}^h}{\varepsilon_h} = \frac{1}{3} \delta_{ij} \quad (20)$$

and

$$\Pi_e = e_{ij} e_{ji} \quad \text{and} \quad \text{III}_e = e_{ij} e_{jk} e_{ki} \quad (21)$$

For the two-component isotropic state and also for the one component state,  $\Pi_a = \Pi_e$ , so that  $A = 1$ , and for these extreme cases, which both lie on the two-component state, eq. (19) satisfies

$$e_{ij} = a_{ij} \quad (22)$$

We may assume, therefore, that this analytical relationship holds along the entire two-component state and suggest an expression for partition of the dissipation tensor  $\varepsilon_{ij}$  for such a state as follows:

$$\varepsilon_{ij} = \frac{1}{4} v \Delta_x \overline{u_i u_j} + \frac{\overline{u_i u_j}}{q^2} \varepsilon_h \quad (23)$$

It can be shown [14], that a Taylor series expansion near the wall for the instantaneous disturbances leads to relations for the asymptotic behaviour of the components of  $\varepsilon_{ij}$  in close agreement with those obtained from eq. (23).

We may follow the same analytical path as outlined above for the treatment of the velocity/pressure gradient correlations, which can be split into the pressure-transport term and the pressure-strain term:

$$\Pi_{ij} = \frac{1}{2} \overline{u_j \frac{\partial p}{\partial x_i}} - \overline{u_i \frac{\partial p}{\partial x_j}} = \underbrace{\frac{1}{\rho} \frac{\partial}{\partial x_j} \overline{p u_i} - \frac{1}{\rho} \frac{\partial}{\partial x_i} \overline{p u_j}}_{\text{pressure-transport}} - \underbrace{\overline{\frac{p}{\rho} \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}}}_{\text{pressure-strain}} \quad (24)$$

In wall-bounded flows the pressure-transport contribution is usually small and we may seek closure for the pressure-strain part by considering the equation for the anisotropy of the stresses in a statistically homogeneous field:

$$\frac{\partial a_{ij}}{\partial t} - \frac{1}{q^2} P_{ij} (a_{ij} - \frac{1}{3} \delta_{ij}) P_{ss} = \frac{\Pi_{ij}}{q^2} - \frac{2\varepsilon_h}{q^2} (a_{ij} - e_{ij}) \quad (25)$$

where  $P_{ij} = \overline{u_i u_k} (\partial U_j / \partial x_k) - \overline{u_j u_k} (\partial U_i / \partial x_k)$ . From this equation we deduce the asymptotic behaviour of  $\Pi_{ij}$  as  $\partial a_{ij} / \partial t \rightarrow 0$  and  $e_{ij} \rightarrow a_{ij}$ , which corresponds to the cases of the axisymmetric distortions when the turbulent stresses approach the limiting states located at the two-component limit:

$$\begin{aligned} (\Pi_{ij})_{2C \text{ iso}} &= a_{ij} P_{ss} - \frac{1}{3} P_{ss} \delta_{ij} P_{ij} \\ (\Pi_{ij})_{1C} &= a_{ij} P_{ss} - \frac{1}{3} P_{ss} \delta_{ij} P_{ij} \end{aligned} \quad (26)$$

This behaviour suggests that the pressure-strain correlations for the two-component state may be approximated as follows:

$$\overline{\frac{p}{\rho} \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}} = \frac{\overline{u_i u_j}}{q^2} P_{ss} - P_{ij} \quad (27)$$

The suggested closure approximations (23) and (27) satisfy the concept of realizability introduced by Schumann [20] and follow closely the data obtained from direct numerical simulations of turbulent wall-bounded flows in the region of the viscous sublayer.

### Determination of the transition criterion

Using the suggested forms for the dissipation and the pressure-strain correlations and for the dissipation rate equation, the transport equations for the two-component disturbances can be written as:

$$\frac{\partial \overline{u_i u_j}}{\partial t} - U_k \frac{\partial \overline{u_i u_j}}{\partial x_k} - 2 \frac{\overline{u_i u_j}}{q^2} (P_k - \varepsilon_h) - \frac{1}{2} \nu \frac{\partial^2 \overline{u_i u_j}}{\partial x_k \partial x_k} \quad (28)$$

and

$$\frac{\partial \varepsilon_h}{\partial t} - U_k \frac{\partial \varepsilon_h}{\partial x_k} - 2 \frac{\varepsilon_h \overline{u_i u_k}}{k} \frac{\partial U_i}{\partial x_k} - \frac{\sqrt{5}}{25} R_\lambda \frac{\varepsilon_h^2}{k} - \frac{1}{2} \nu \frac{\partial^2 \varepsilon_h}{\partial x_k \partial x_k} \quad (29)$$

where  $P_k = P_{ss}/2$ .

If we consider transition of the flow in the flat plate boundary layer, in the way proposed by Taylor [21], then eq. (28) immediately suggests stability towards small disturbances if the production is balanced by the dissipation:

$$P_k \cong \varepsilon_h \quad (30)$$

The equilibrium constraint leads to the equations for the stresses:

$$\frac{\partial \overline{u_i u_j}}{\partial t} - U_k \frac{\partial \overline{u_i u_j}}{\partial x_k} - \frac{1}{2} \nu \frac{\partial^2 \overline{u_i u_j}}{\partial x_k \partial x_k} \quad (31)$$

which are of boundary layer character and do not allow amplification of disturbances in the boundary layer [30]. In connection with this issue, it is interesting that the results of Becker [22] on laser-Doppler measurements in the transitional boundary layer developing in the presence of natural disturbances corroborate the findings mentioned above. Inserting eq. (30) into the dissipation rate eq. (29):

$$\frac{\partial \varepsilon_h}{\partial t} - U_k \frac{\partial \varepsilon_h}{\partial x_k} - \underbrace{2 \frac{\sqrt{5}}{25} R_\lambda \frac{\varepsilon_h^2}{k}}_0 - \frac{1}{2} \nu \frac{\partial^2 \varepsilon_h}{\partial x_k \partial x_k} \quad (32)$$

to insure that  $\varepsilon_h > 0$

and specifying that the dissipation rate is always positive,  $\varepsilon_h > 0$ , and at the critical point follows the energy  $k$  (as emerges from the work of Kolmogorov [23]), we deduce the transition criterion in terms of the Reynolds number based on the intensity and the length scale of the disturbances:

$$(R_\lambda)_{crit} = 10\sqrt{5} \quad (33)$$

Thus, the derived transition criterion suggests the permissible magnitudes for the intensity and the length scale of disturbances  $R_\lambda > (R_\lambda)_{crit}$  guarantee that equality (30) holds, ( $P_k = \varepsilon_h$  with  $\varepsilon_h > 0$ ), and therefore maintenance of the laminar flow regime in the flat plate boundary layer.

### Analysis of the transition process in wall-bounded flows

The governing equations describing the behaviour of the laminar to turbulence transition process, eqs. (28) and (29), are identical with those for fully developed turbulent flows in the region of the viscous sublayer close to the wall. We may attempt, therefore, to use the data for these flows to extract information about the functional dependence of  $R_\lambda$  in terms of the Reynolds number based on the global flow parameters in order to demonstrate the validity of the derived criterion (33) for the onset of turbulence in wall-bounded flows.

Figure 4 shows the data for  $R_\lambda$ , of fully developed turbulent channel flow, averaged over the cross-section, versus the Reynolds number,  $Re_\tau = Hu_\tau/\nu$ , based on

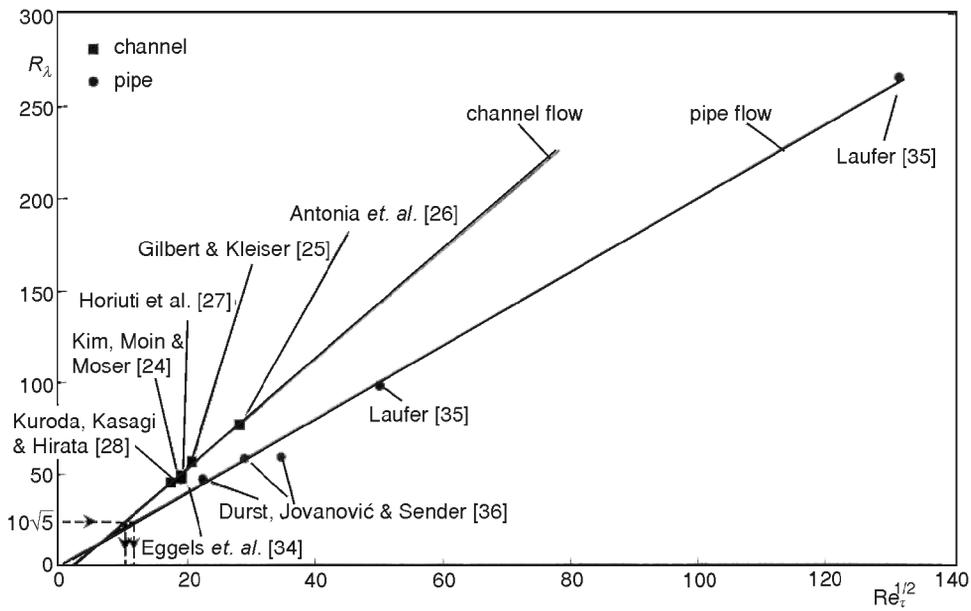


Figure 4. Cross plot of  $R_\lambda$  versus  $Re_\tau^{1/2}$  for fully developed turbulent pipe and channel flows at low Reynolds numbers

the full height,  $H$ , of the channel and the friction velocity,  $u_\tau$ . These data were calculated from databases of direct numerical simulations (DNS) from Kim, Moin, and Moser [24], Gilbert and Kleiser [25], Antonia et al. [26], Horiuti et al. [27], and Kuroda, Kasagi, and

Hirata 28 . The plotted data closely follow the expected behaviour, proportional to  $Re_\tau^{1/2}$ , based on the evidence that, near the channel centreline, turbulence intensities scale with the wall variables (Durst *et al.* 29 ). The least-squares fit of the data from fig. 4 yields:

$$R_\lambda = 2.971 \frac{Hu_\tau}{\nu}^{1/2} - 6.618 \quad (34)$$

By extrapolating relation (34) towards the transition criterion (33) using the friction coefficient valid for laminar flow conditions, we obtain the critical Reynolds number based on the full channel height and the centreline velocity as:

$$(Re)_{crit} = 2260 \quad (35)$$

This result is in close agreement with all accumulated observations available from either numerical simulations or flow visualization experiments (see, for example, 31, 32, 33 ).

We may consider, in an analogous way, transition in a pipe flow. For this flow there exists only one set of published data from full numerical simulations, reported by Eggels *et al.* 34 . An effort was made, therefore, to deduce the required information using the published experimental data on fully developed turbulent pipe flow from Laufer 35 and Durst, Jovanović, and Sender 36 . These authors provided information for the mean flow and the intensities of all three fluctuating velocity components. To determine  $R_\lambda$  from these data the mean energy dissipation rate was calculated from the mean energy production assuming flow equilibrium. The computed data for  $R_\lambda$  are displayed in fig. 4 in the same form as for the channel flow with the exception that the Reynolds number  $Re_\tau = Du_\tau/\nu$  is defined in terms of the pipe diameter,  $D$ . A least-squares fit through the pipe data gives:

$$R_\lambda = 1.996 \frac{Du_\tau}{\nu}^{1/2} - 0.108 \quad (36)$$

If we extrapolate eq. (36) in the same way as for the channel flow, the critical Reynolds number based on the pipe diameter and the bulk velocity emerges as:

$$(Re)_{crit} = 1930 \quad (37)$$

The value obtained is in good agreement with the results deduced from flow visualization experiments by Reynolds (1883) 37 ,  $1900 < (Re)_{crit} < 2000$ , and many other experimental studies reviewed in the revised and updated book by Monin and Yaglom 38 .

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